

Complex Numbers

Introduction -

- * Definition - The numbers of the form $x + jy$ is set to be Complex numbers

And given by $Z = x + jy$

Where, x = Real Part

y = Imaginary Part

and $j = \sqrt{-1}$, $j^2 = -1$, $j^3 = -j$

- * Combination of Real part and imaginary part forms a Complex number.

- * j is known as imaginary unit.

- * Complex number is generally denoted by

Z i.e., $Z = x + jy$

Complex Conjugate

- $Z = x + jy$ and $\bar{Z} = x - jy$ are known as Complex Conjugate of each other.

e.g. $2+j3 \rightarrow 2-j3$

$-4-j5 \rightarrow -4+j5$

• Modulus of Complex number -

Complex number $Z = x+jy$ then its mod value is denoted by $|Z|$ and given by

$$|Z| = \sqrt{x^2+y^2} = \sqrt{(\text{R.P.})^2 + (\text{I.M.})^2}$$

• Cartesian form - If Complex number Z is represented by variables x and y , then it is called as Cartesian form of complex number

e.g., $Z = x+jy$, $2+j3$, $5-j8$, etc.

* Polar form of a Complex Number - *

If the Complex number Z is represented by r and θ , then it is called polar form of a complex number

$$Z = x+jy$$

How to find polar form

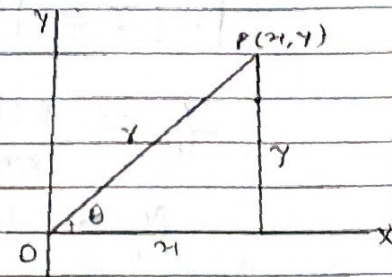
We have, $Z = x+jy$ Put $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

Teacher's Signature.....

$$\therefore Z = r \cos \theta + jr \sin \theta$$

$$Z = r(\cos \theta + j \sin \theta)$$

$$Z = r e^{j\theta}$$



std. formula:

since $e^{j\theta} = \cos \theta + j \sin \theta$

where, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$

$$Z = r(\cos \theta + j \sin \theta)$$

is called polar form of complex number.

r is called modulus or absolute value.

written as $|\text{mod}(x+jy)|$ or $|x+jy|$

θ is called amplitude or argument.

written as $|\text{amp}(x+jy)|$ or $|\text{arg}(x+jy)|$

r, θ value find,

$$x^2 = r^2 \cos^2 \theta, y^2 = r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$x^2 + y^2 = r^2 (1)$$

Teacher's Signature.....

$$\therefore r = \sqrt{x^2 + y^2}$$

$$\frac{y}{r} = \frac{r \sin \theta}{r \cos \theta}$$

$$\frac{y}{x} = \tan \theta$$

$$\therefore \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

* Argument of complex numbers -

Step-I $z = a + ib$

$$\tan \theta = \left| \frac{b}{a} \right|$$

Step-II Locate quadrant of z

if ' z ' lies in I-quadrant $\Rightarrow \arg(z) = \theta$

if ' z ' lies in II-quadrant $\Rightarrow \arg(z) = \pi - \theta$

if ' z ' lies in III-quadrant $\Rightarrow \arg(z) = \theta - \pi$

if ' z ' lies in IV-quadrant $\Rightarrow \arg(z) = -\theta$

Note:- $[-\pi < \arg(z) \leq \pi]$ Principal value
 $\downarrow$ $\downarrow$
 -180° 180°

Example- $z = 1 - \sqrt{3}i$ $\arg(z) = ?$

Solution $\Rightarrow a = 1, b = -\sqrt{3}$

$$\tan \theta = \left| \frac{-\sqrt{3}}{1} \right| = \sqrt{3}$$

Teacher's Signature.....

$$\tan \theta = \sqrt{3}$$

$$\theta = 60^\circ$$

$$\arg(z) = -60^\circ \text{ or } -\frac{\pi}{3}$$

* Properties - Argument

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pm 2n\pi$

(ii) $\arg(z_1^n) = n \arg(z_1) \pm 2n\pi$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \pm 2n\pi$

(iv) $\arg(z) = -\arg(\bar{z})$

OR $\arg(\bar{z}) = -\arg(z)$

$(1 - \sqrt{3}i)(1 + \sqrt{3}i) = 1 - (\sqrt{3}i)^2 = 1 - (-3) = 1 + 3 = 4$

Teacher's Signature.....

Complex Integration

Let $f(z)$ be an analytic function in any domain then Integration of $f(z)$ along any curve joining the points $z=a$ To $z=b$

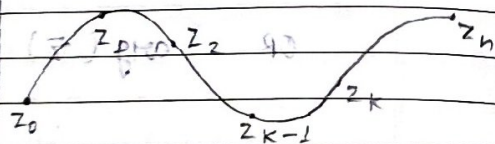
$$\int_a^b f(z) dz$$

Complex line Integration

Let L be a realizable arc defined by

$$z = z(t) = x(t) + iy(t) \quad (iii) \\ a \leq t \leq b$$

and now divide the arc L into n small arcs by the points



$$a = z_0, z_1, \dots, z_{k-1}, z_k, \dots, z_n = b$$

Then Complex Integration of $f(z)$ over arc L is defined by

$$\int_L f(z) dz = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(\xi_k) (z_k - z_{k-1})$$

Teacher's Signature.....

Example - Evaluate the following integrals.

(i) $\int_L dz$ (ii) $\int_L |dz|$ (iii) $\int_L z dz$

Let $a = z_0, b = z_n$

Solⁿ $\Rightarrow$ (i) $\int_L dz = \lim_{n \rightarrow \infty} \sum_{k=1}^n (z_k - z_{k-1}) \quad \because f(z) = 1$

$$= \lim_{n \rightarrow \infty} [(z_1 - z_0) + (z_2 - z_1) + \dots + (z_n - z_{n-1})]$$

$$= \lim_{n \rightarrow \infty} [z_1 - z_0 + z_2 - z_1 + z_3 - z_2 + \dots + z_n - z_{n-1}]$$

$$= \lim_{n \rightarrow \infty} [z_n - z_0]$$

$$\int_L dz = b - a$$

Ans

(ii) $\int_L |dz|$

again $f(z) = 1$

$$\int_L |dz| = \lim_{n \rightarrow \infty} \sum_{k=1}^n |z_k - z_{k-1}|$$

$$= \lim_{n \rightarrow \infty} [|z_1 - z_0| + |z_2 - z_1| + \dots + |z_n - z_{n-1}|]$$

Teacher's Signature.....

$$= \lim_{n \rightarrow \infty} [\text{chord } z_1 z_0 + \text{chord } z_2 z_1 + \dots + \text{chord } z_n z_{n-1}]$$

$$= \lim_{n \rightarrow \infty} [\text{arc } z_1 z_0 + \text{arc } z_2 z_1 + \dots + \text{arc } z_n z_{n-1}]$$

$$= \text{arc length } L$$

Example-1) Prove that $\int_C \frac{dz}{z} = 2\pi i$ where C is unit circle $|z|=1$

Proof - Given integral

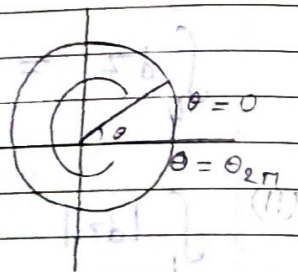
$$\int_C \frac{dz}{z} = 2\pi i$$

where C is circle $|z|=1$

$$z = r \cos \theta + j r \sin \theta$$

$$z = r(\cos \theta + j \sin \theta)$$

$$z = r e^{j\theta} \quad dz = r j e^{j\theta} d\theta$$



$$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{j r e^{j\theta} d\theta}{r e^{j\theta}}$$

$$= j \int_0^{2\pi} d\theta$$

$$= j [2\pi - 0]$$

Teacher's Signature.....

$$\int_C \frac{1}{z} dz = 2\pi i$$

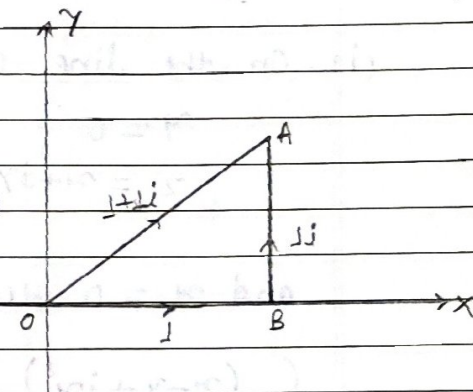
Example-(2) Evaluate integral $\int_0^{1+j} (x-y+jx^2) dz$

(a) along the straight line from $z=0$ to $z=1+j$

(b) along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+j$

Solution $\Rightarrow \because z = x + jy$
 $\therefore dz = dx + jdy$

(a) Let OA be the line from $z=0$ to $z=1+j$



Then, on OA $y=x$

$$z = x + jy$$

$$\therefore y=x$$

$$z = x + jx$$

$$= (1+j)x$$

$$dz = (1+j) dx$$

$$\int_{OA} (x-y+jx^2) dz = \int_0^1 (x-x+jx^2) (1+j) dx$$

Teacher's Signature.....

$$= \int_0^1 (1+j)jx^2 dx$$

$$= (-1+j) \int_0^1 x^2 dx$$

$$= (-1+j) \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{-1+j}{3}$$

(b) Integral about OB and BA

(i) On the line OB real axis

$$y=0$$

$$z=x+jy \Rightarrow z=x$$

$$dz=dx$$

and $x=0$ to $x=1$

$$\int_{OB} (x-y+jx^2) dz = \int_0^1 (x-0+jx^2) dx$$

$$= \int_0^1 x dx + j \int_0^1 x^2 dx$$

$$= \left[\frac{x^2}{2} \right]_0^1 + j \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} + \frac{j}{3}$$

Teacher's Signature.....

(ii) BA is the line parallel to imaginary axis
from $z=1$ to $z=1+j$

$$x=1 \quad z=1+jy$$

$$dz = jdy$$

$$\int_{BA} (x-y+jx^2) dz$$

$$= \int_0^1 (1-y+j) jdy$$

$$= j \left\{ [(1+j)y]_0^1 - \left[\frac{y^2}{2} \right]_0^1 \right\}$$

$$= j \left\{ (1+j) - \frac{1}{2} \right\}$$

$$= j \left\{ (1+j) - \frac{1}{2} \right\}$$

$$= j \left\{ \frac{1}{2} + j - \frac{1}{2} \right\} = j^2 = -1$$

$$j^2 = -1$$

$$j^2 = -1$$

negative value of
negative

$$j^2 - j^2 = -1$$

Teacher's Signature.....

Cauchy's Theorem

Let $f(z)$ be an analytic function and let $f'(z)$ be continuous at each point with in and on a closed contour C , then

$$\int_C f(z) dz = 0$$

Proof:- Given that $f(z)$ is analytic in and on closed contour C and let D denote a closed domain consisting of all points with in and on C assume that

$$f(z) = u(x, y) + iv(x, y)$$

$$\int_C f(z) dz = \int_C u(x, y) + iv(x, y) (dx + idy)$$

$$\int_C f(z) dz = \int_C (u dx - v dy + i) \int_C u dy + v dx \quad \text{--- (1)}$$

$$f'(z) = u_x + iv_x$$

$$\therefore u_x = v_y$$

$$v_x = -u_y$$

$$f'(z) = v_y - iu_y$$

by Cauchy Riemann equation

Since $f(z)$ is analytic in D so there have exist partial derivatives u_x, u_y, v_x and v_y and by Green's theorem

Theorem from (1)

$$\int_C f(z) dz = \int_C \frac{\partial y}{\partial x} - \frac{\partial v}{\partial x} \iint_D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$+ i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

$\therefore$ by Cauchy Riemann equation

$$= \int_C \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \right) dx dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$= 0 + 0 = 0$$

$$\therefore \int_C f(z) dz = 0$$

Cauchy-Goursat Theorem

If a function $f(z)$ is analytic and single valued function inside and on a simple closed contour C , then

$$\int_C f(z) dz = 0$$

Proof:- For the proof of the theorem firstly we require to prove the lemmas.

Lemma:- $f(z)$ is analytic function with in and on a closed contour C . Then for any given $\epsilon > 0$ it is always possible to divide the region inside C into number of meshes either complete square C_n or partial square D_n such that within each mesh there exists a point Z_0 such that

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \epsilon |z - z_0| \quad (1)$$

for every $z (\neq z_0)$ within σ_1 on the mesh.

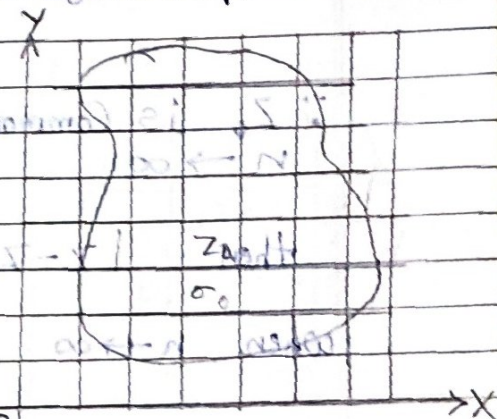
Proof:- Given region closed contour C divided up into a number of sub-regions by a network of lines parallel to the real and imaginary axes, and let this divided the inside of C into a number of squares $C_1, C_2, \dots, C_m$, say, and a partial squares $D_1, D_2, \dots, D_n$ say part of whose

Teacher's Signature.....

Boundary is also part to boundary of C let l is distance between two consecutive line.

Let lemma is true for any point Z_0 is some square and let lemma is false in any square. Then we sub divide such square where lemma is false in four equal four part σ_0 is denote the mesh where lemma is false. then σ_0 sub divide in four equal square let and let again lemma is false at Z_0 in σ_1 again we quadrisection σ_1 and proceed in this manner then there are two distinct possibilities.

- (1) This process may terminate after a finite number of steps and at the end we obtain the desired result



- (2) This process may continue indefinitely in this case we get a nested sequence of limit $\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_n$ each contained in the preceding one such that in none of these there exists a Z_0 with the desired property.

There is no a point Z_0 exists where lemma is true

Teacher's Signature.....

But given that $f(z)$ is analytic in closed region D consisting of points inside and on the closed contour C . Thus it follows that $f(z)$ is differentiable at any point $z \in D$. Hence for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{--- (2)}$$

$$\forall |z - z_0| < \delta$$

$\therefore z_0$ is common point in all σ_n and when $n \rightarrow \infty$

$$\text{then } |z - z_0| \rightarrow 0$$

when $n \rightarrow \infty$

$$z \rightarrow z_0 \quad \forall z \in \sigma_n$$

hence z_0 is limit point of D and since D is closed, $z_0 \in D$.

hence from (2)

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \varepsilon |z - z_0|$$

$$\forall z \in \sigma_n$$

which is contradiction of statement (1)

Teacher's Signature.....

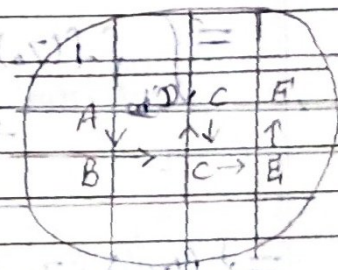
Hence lemma is proved.

Proof of the main theorem.

We divide up interior of C into complete squares $C_1, C_2, \dots, C_N$ and $D_1, D_2, \dots, D_N$ partial squares by network of line parallel to real and imaginary axis. Then we can write.

$$\int_C f(z) dz = \sum_{m=1}^N \int_{C_m} f(z) dz + \sum_{n=1}^N \int_{D_n} f(z) dz \quad \text{--- (3)}$$

Now consider two adjacent square ABCD and DCEF with common side CD is describe



from $C \rightarrow D$ in ABCD and from $D \rightarrow C$ in DCEF

Therefore two integral along CD will cancel each other, by the same arguments we find all the integral.

Cancelled except those which are taken along two parts of the contour C itself.

Teacher's Signature.....

Case-I Now Consider a integral curve

$$\int_{C_m} f(z) dz$$

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + (z-z_0)\eta(z)$$

$$\text{where } |\eta(z)| \leq \epsilon \quad (4)$$

$$\int_{C_m} f(z) dz = \int_{C_m} f(z_0) + (z-z_0)f'(z_0) dz + \int_{C_m} (z-z_0)\eta(z) dz$$

$$= \int_{C_m} \{f(z_0) + zf'(z_0) - z_0 f'(z_0)\} dz + \int_{C_m} \{z\eta(z) - z_0\eta(z)\} dz$$

$$= \{f(z_0) - z_0 f'(z_0)\} \int_{C_m} dz + f'(z_0) \int_{C_m} z dz + \int_{C_m} (z-z_0)\eta(z) dz$$

$\therefore C_m$ is closed

$$\int_{C_m} dz = 0 \text{ and } \int_{C_m} z dz = 0$$

$$\int_{C_m} f(z) dz = \int_{C_m} (z-z_0)\eta(z) dz$$

$$\left| \int_{C_m} f(z) dz \right| \leq \int_{C_m} |z-z_0| |\eta(z)| |dz|$$

$$\therefore |z-z_0| < \sqrt{2} l_m$$

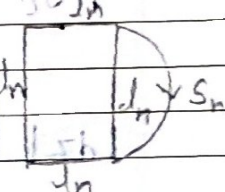
$$|\eta(z)| < \epsilon$$

$$< \sqrt{2} l_m \cdot \epsilon \int_{C_m} dz$$

$$\left| \int_{C_m} f(z) dz \right| < \sqrt{2} l_m \epsilon \cdot 4 l_m \quad (5)$$

Case-II Now Consider irregular sub regions D_n . In this case the length of the contour D_n is not greater than $4l_n + S_n$

where S_n is length of curved part of D_n

$$\left| \int_{D_n} (z-z_0)\eta(z) dz \right| \leq \int_{D_n} |z-z_0| |\eta(z)| dz$$


$$\left| \int_{D_n} (z-z_0)\eta(z) dz \right| < \epsilon \sqrt{2} l_n (4l_n + S_n)$$

from (3)

$$\left| \int_C f(z) dz \right| \leq \sum_{m=1}^N \left| \int_{C_m} f(z) dz \right| + \sum_{n=1}^N \left| \int_{D_n} f(z) dz \right|$$

$$\left| \int_C f(z) dz \right| < \sum_{m=1}^M \left| \int_{C_m} (z-z_0) \eta(z) dz \right| + \sum_{n=1}^N \left| \int_{D_n} (z-z_0) \eta(z) dz \right|$$

$$\left| \int_C f(z) dz \right| < 4\sqrt{\epsilon} \sum (\sqrt{J_m^2 + J_n^2}) + \epsilon \sqrt{2} J_n \leq S_n$$

$\therefore 4\sqrt{\epsilon} \sum (\sqrt{J_m^2 + J_n^2}) + \epsilon \sqrt{2} J_n \leq S_n$
are constant quantity so we can denote K

$$\left| \int_C f(z) dz \right| < K\epsilon \quad (6)$$

$\therefore \epsilon$ is arbitrary
 $\epsilon \rightarrow 0$

it follows from (6) that

$$\int_C f(z) dz = 0$$

The theorem is thus completely established.

21, 19, 15, 13

P-176

Cauchy Integral Formula

Let $f(z)$ is analytic with in and on a closed contour C and let z_0 be any point with in C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - z_0}$$

Proof - Given that $f(z)$ is analytic with in and on a closed contour C now we describe a Circle γ defined by the equation

$$|z - z_0| = \rho \text{ and } \rho < r$$

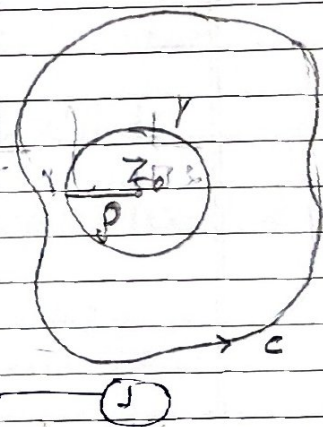
the function

$$\phi(z) = \frac{f(z)}{z - z_0}$$

$$\int_C \phi(z) dz = \int_\gamma \phi(z) dz$$

$$\frac{1}{2\pi i} \int_C \phi(z) dz = \frac{1}{2\pi i} \int_\gamma \phi(z) dz$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$



Since $f(z)$ is analytic in C therefore $f(z)$ is continuous in C . then by definition of continuity at $z = z_0$ for given $\varepsilon > 0$ there exist $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(z_0) + f(z_0)}{z - z_0} dz$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(z_0)}{z - z_0} dz + \frac{1}{2\pi i} \int_{\gamma} \frac{f(z_0)}{z - z_0} dz$$

$$+ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z_0)}{z - z_0} dz \quad (2)$$

writing $z - z_0 = \rho e^{i\theta}$

$$dz = i \rho e^{i\theta} d\theta$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z_0)}{z - z_0} dz = \frac{f(z_0)}{2\pi i} \int_0^{2\pi} \frac{i \rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$

$$= \frac{f(z_0) \cdot 2\pi}{2\pi}$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z_0)}{z - z_0} dz = f(z_0)$$

from (2)

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(z_0)}{z - z_0} dz + f(z_0)$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) \right| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(z_0)}{z - z_0} dz \right|$$

$$z - z_0 = \rho e^{i\theta} \\ dz = i \rho e^{i\theta} d\theta \\ = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z) - f(z_0)}{\rho e^{i\theta}} i \rho e^{i\theta} d\theta \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z) - f(z_0)| d\theta$$

$$\therefore |f(z) - f(z_0)| < \varepsilon \\ \leq \frac{1}{2\pi} \varepsilon \int_0^{2\pi} d\theta$$

$$< \frac{1}{2\pi} \varepsilon \cdot 2\pi$$

$$\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz - f(z_0) \right| < \epsilon$$

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz = f(z_0)$$

$$\Rightarrow \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz = f(z_0)$$

P-183

M31, 17

The Derivative of an Analytic Function

Theorem— Let $f(z)$ be analytic function with in and on boundary C of a simply connected region D and let z_0 be any points with in C then

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$$

Proof— Given that $f(z)$ be analytic function, with in and on boundary C of a simply connected region D and let z_0 be any point in D and let z_0+h be a neighbouring point of z_0 then by Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$$

Teacher's Signature.....

Date 6/1/1

$$f(z_0+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-(z_0+h)} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0-h} dz$$

$$f(z_0+h) - f(z_0) = \frac{1}{2\pi i h} \left[\int_C \frac{f(z)}{z-z_0-h} dz - \int_C \frac{f(z)}{z-z_0} dz \right]$$

$$= \frac{1}{2\pi i h} \left[\int_C f(z) \left[\frac{1}{(z-z_0-h)} - \frac{1}{(z-z_0)} \right] dz \right]$$

$$= \frac{1}{2\pi i h} \int_C f(z) \frac{z-z_0 - (z-z_0-h)}{(z-z_0-h)(z-z_0)} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0-h)(z-z_0)} dz$$

$$\left[\frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0-h)(z-z_0)} dz \right]$$

$$= \left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0-h)(z-z_0)} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz \right]$$

$$= \left[\frac{1}{2\pi i} \int_C f(z) \left\{ \frac{1}{(z-z_0-h)(z-z_0)} - \frac{1}{(z-z_0)^2} \right\} dz \right]$$

Teacher's Signature.....

$$= \left[\frac{1}{2\pi i} \int_C f(z) \frac{z - z_0 - z + z_0 + h}{(z - z_0 - h)(z - z_0)^2} dz \right] \\ = \left[\frac{h}{2\pi i} \int_C \frac{f(z)}{(z - z_0 - h)(z - z_0)^2} dz \right] \quad (1)$$

Now describe a circle γ with centre z_0 and radius ρ such that γ lies entirely within C . Then by the Cauchy-Goursat theorem

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2 (z - z_0 - h)} dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2 (z - z_0 - h)} dz \quad (2)$$

let us choose h so small such that $z_0 + h$ is lie inside γ

$$\text{and } h < \frac{1}{2}\rho$$

$$\text{and } |z - z_0| = \rho \quad \text{and } |e^{i\theta}| = 1$$

$$|z - z_0 - h| \geq |z - z_0| - |h|$$

$$\geq \rho - \frac{1}{2}\rho$$

$$|z - z_0 - h| > \frac{1}{2}\rho$$

from (1) and (2)

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz \right|$$

$$= \left| \frac{h}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0 - h)(z - z_0)^2} dz \right|$$

$$\leq \frac{|h|}{2\pi} \int_\gamma \frac{|f(z)|}{|z - z_0 - h| |z - z_0|^2} |dz|$$

$$\therefore |z - z_0| < \rho$$

$$|z - z_0 - h| < \frac{1}{2}\rho$$

and $f(z)$ is analytic $\Rightarrow f(z)$ is bounded in D . So $|f(z)| \leq M$ where M is constant

$$\leq \frac{|h|}{2\pi} \int_\gamma \frac{M |dz|}{\rho^2 \frac{1}{2}\rho}$$

$$\leq \frac{|h|}{2\pi} \frac{M}{\rho^2 \frac{1}{2}\rho} \int_\gamma |dz|$$

$$\therefore \int_\gamma |dz| = 2\pi\rho$$

$$\leq \frac{|h|}{2\pi} \frac{M}{\rho^2} \times 2\pi\rho$$

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{2|h|M}{\rho}$$

$\therefore h$ is very small
take $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{-f(z)}{(z-z_0)^2} dz = 0$$

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$$

Which is Cauchy's integral formula for $f'(z_0)$ at points within C .

P-185

Higher Order Derivatives

Statement - Let $f(z)$ is analytic within and on the boundary C of a Simply Connected region D and let z_0 be any point within C . Then derivatives of all orders are analytic and given by

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Proof - Given that $f(z)$ is analytic in and on the boundary C of a Simply Connected region D and let z_0 and z_0+h are two neighbour points in D . Then by Cauchy integral formula

$$\begin{aligned} \frac{f'(z_0+h) - f'(z_0)}{h} &= \frac{1}{h} \left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0-h)^2} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz \right] \\ &= \frac{1}{h} \left[\frac{1}{2\pi i} \int_C f(z) \left[\frac{1}{(z-z_0-h)^2} - \frac{1}{(z-z_0)^2} \right] dz \right] \\ &= \frac{1}{2\pi i h} \left[\int_C \frac{f(z) [(z-z_0)^2 - (z-z_0-h)^2]}{(z-z_0-h)^2 (z-z_0)^2} dz \right] \end{aligned}$$

$$= \frac{1}{2\pi i h} \int_C \frac{f(z) [(z-z_0)^2 - (z-z_0)^2 - h^2 + 2h(z-z_0)]}{(z-z_0-h)^2 (z-z_0)^2} dz$$

$$= \frac{1}{2\pi i h} \int_C \frac{f(z) (2(z-z_0) - h)}{(z-z_0-h)^2 (z-z_0)^2} dz$$

$$\frac{f'(z_0+h) - f'(z_0)}{h} = \frac{2!}{2\pi i} \int_C \frac{((z-z_0) - \frac{h}{2}) f(z)}{(z-z_0-h)^2 (z-z_0)^2} dz$$

$$\frac{f'(z_0+h) - f'(z_0)}{h} = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

$$= \frac{2!}{2\pi i} \int_C \frac{(z-z_0 - \frac{h}{2}) f(z)}{(z-z_0-h)^2 (z-z_0)^2} dz$$

$$= \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

$$= \frac{2!}{2\pi i} \int_C \left[\frac{(z-z_0 - \frac{h}{2})}{(z-z_0-h)^2 (z-z_0)^2} - \frac{1}{(z-z_0)^3} \right] f(z) dz$$

$$= \frac{2!}{2\pi i} \int_C \frac{(z-z_0)^2 + \frac{h}{2}(z-z_0) - (z-z_0-h)^2}{(z-z_0-h)^2 (z-z_0)^3} dz$$

$$= \frac{2!}{2\pi i} \int_C \frac{[(z-z_0)^2 - \frac{h}{2}(z-z_0) - (z-z_0)^2 + 2h(z-z_0) - h^2]}{(z-z_0-h)^2 (z-z_0)^3} dz$$

Teacher's Signature.....

$$= \frac{2!}{2\pi i} \int_C \frac{\frac{3}{2} h(z-z_0) - h^2}{(z-z_0-h)^2 (z-z_0)^3} dz$$

Let γ is circle in C i.e. $|z-z_0| = \rho$

$$\left| \frac{f'(z_0+h) - f'(z_0)}{h} - \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz \right|$$

$$= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{\left\{ \frac{3}{2} h(z-z_0) - h^2 \right\}}{(z-z_0-h)^2 (z-z_0)^3} dz \right|$$

$$\leq \frac{2|h|}{2\pi} \int_{\gamma} \frac{\frac{3}{2} |z-z_0| + |h|}{|z-z_0-h|^2 |z-z_0|^3} |f(z)| |dz|$$

$$|h| < \frac{1}{2} \rho$$

$$\therefore |z-z_0| = \rho$$

$$|z-z_0-h| \geq \frac{1}{2} \rho$$

$$\leq \frac{2 \frac{1}{2} |h|}{2\pi} \int_{\gamma} \frac{\frac{3}{2} \rho + \frac{1}{2} |h|}{\frac{1}{4} \rho^2 \cdot \rho^3} |f(z)| |dz|$$

$$\therefore |f(z)| = M$$

$$\int_{\gamma} |dz| = \text{Perimeter of circle} \\ \gamma = 2\pi\rho$$

$$\leq \frac{2|h|}{2\pi} \cdot \frac{\frac{3}{2} \rho + |h|}{\rho^3 \cdot \frac{1}{4} \rho^2} \int_{\gamma} |dz|$$

$$\leq \frac{2|h| \cdot \frac{3}{4} \rho + |h|}{2\pi \rho^3 \cdot \frac{1}{4} \rho^2} \cdot 2\pi \rho$$

length $|h| \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{f'(z_0+h) - f'(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz = 0$$

$$\lim_{h \rightarrow 0} \frac{f'(z_0+h) - f'(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

$$f''(z_0) = \frac{1}{\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

which is true for $n=2$

now let theorem is true $n=m$

$$f^m(z_0) = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{m+1}} dz$$

$$f^m(z_0) = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{m+1}} dz$$

now we will prove for $n=m+1$

$$\frac{f^{m+1}(z_0+h) - f^{m+1}(z_0)}{h} = \frac{1}{2\pi i} \int_C \left[\frac{1}{(z-z_0-h)^{m+1}} - \frac{1}{(z-z_0)^{m+1}} \right] f(z) dz$$

Teacher's Signature.....

$$= \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[\frac{1}{\left(1 - \frac{h}{z-z_0}\right)^{m+1}} - 1 \right] f(z) dz$$

$$= \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[\left(1 - \frac{h}{z-z_0}\right)^{-(m+1)} - 1 \right] f(z) dz$$

$$= \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[1 + \frac{(m+1)h}{(z-z_0)} + \frac{h^2}{2! (z-z_0)^2} + \dots - 1 \right] f(z) dz$$

$$= \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[\frac{(m+1)h}{(z-z_0)} + \frac{h^2}{2! (z-z_0)^2} + \dots \right] f(z) dz$$

Now taking limit $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{f^{m+1}(z_0+h) - f^{m+1}(z_0)}{h} = \lim_{h \rightarrow 0} \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[\frac{(m+1)h}{(z-z_0)} + \frac{h^2}{2! (z-z_0)^2} + \dots \right] f(z) dz$$

$$= \lim_{h \rightarrow 0} \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[1 + \frac{h(m+1)}{z-z_0} + \frac{h^2(m+1)(m+2)}{2! (z-z_0)^2} + \dots \right] f(z) dz$$

$$= \lim_{h \rightarrow 0} \frac{m!}{h 2\pi i} \int_C \frac{1}{(z-z_0)^{m+1}} \left[1 + \frac{h(m+1)}{z-z_0} + \frac{h^2(m+1)(m+2)}{2! (z-z_0)^2} + \dots \right] f(z) dz$$

now taking $h \rightarrow 0$

Teacher's Signature.....

$$\lim_{h \rightarrow 0} \frac{f^{(m)}(z_0+h) - f^{(m)}(z_0)}{h} = \frac{(m+1)!}{2\pi i}$$

$$\int_C \frac{f(z) dz}{(z-z_0)^{m+2}}$$

$$f^{(m+1)}(z_0) = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{m+2}} dz$$

which is true for $n = m+1$

hence by Mathematical Induction Jacob's Theorem is true for $n = n$.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Also we see from this result that $f^{(n)}(z_0)$ is an analytic function of z .

This implies that derivatives of $f(z)$ of all orders are analytic if $f(z)$ is analytic.

P-196

17, 14, 12, 11

Morera's Theorem

Th^m - Let $f(z)$ be a Continuous in a Simply Connected domain D and let for every closed Contour C in the domain D ,

$$\int_C f(z) dz = 0$$

Then $f(z)$ is analytic in D .

Proof - Given that $f(z)$ is Continuous in D then now we define a function $F(z)$ following

$$F(z) = \int_{z_0}^z f(\xi) d\xi \quad \text{--- (1)}$$

Let $z+h$ is neighbourhood point of z

$$F(z+h) = \int_{z_0}^{z+h} f(\xi) d\xi \quad \text{--- (2)}$$

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\xi) d\xi - \int_{z_0}^z f(\xi) d\xi$$

$$= \int_{z_0}^{z+h} f(\xi) d\xi + \int_z^{z_0} f(\xi) d\xi$$

$$F(z+h) - F(z) = \int_z^{z+h} f(\xi) d\xi$$

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_z^{z+h} f(\xi) d\xi$$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} f(\xi) d\xi - f(z)$$

$$= \frac{1}{h} \int_z^{z+h} f(\xi) d\xi - \frac{1}{h} \int_z^{z+h} f(z) d\xi$$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \left| \int_z^{z+h} \{f(\xi) - f(z)\} d\xi \right|$$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{1}{|h|} \int_z^{z+h} |f(\xi) - f(z)| d\xi$$

$\therefore f$ is continuous at each point z in D and $\xi, z \in D$

Teacher's Signature.....

then $|f(\xi) - f(z)| < \epsilon$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| < \frac{1}{|h|} \int_z^{z+h} \epsilon d\xi$$

$$< \frac{1}{|h|} \epsilon |z+h-z|$$

$$\frac{F(z+h) - F(z)}{h} - f(z) < \epsilon$$

$$\lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

$$\boxed{F'(z) = f(z)}$$

hence $F(z)$ is differentiable.

$\therefore f(z)$ is analytic in D .

Teacher's Signature.....

Poisson's Integral Formula

Theorem - Let $f(z)$ is analytic in region $|z| < \rho$ and let $z = re^{i\theta}$ be any point of this region. Then

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr \cos(\theta - \phi) + r^2} d\phi$$

where R is any number such that $0 < R < \rho$

Proof - Let C denote the circle

$|z| = R$ such that $r < R < \rho$

and $z = re^{i\theta}$ be any point of the region

$|z| < \rho$

Now by Cauchy integral formula for $z \in D$.

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{--- (1)}$$

$|z| = R$ is Circle of radius R

inverse point of z

Teacher's Signature.....

circle with respect to the $|z| = R$

$$\text{is } \frac{R^2}{\bar{z}} \because z\bar{z} = R^2$$

$\therefore f(\zeta)$ is analytic function

$$\Rightarrow \frac{f(\zeta)}{\zeta - \frac{R^2}{\bar{z}}}$$

is also analytic in C

then by Cauchy-Goursat theorem formula

$$\int_C \frac{f(\zeta)}{\zeta - \frac{R^2}{\bar{z}}} d\zeta = 0 \quad \text{--- (2)}$$

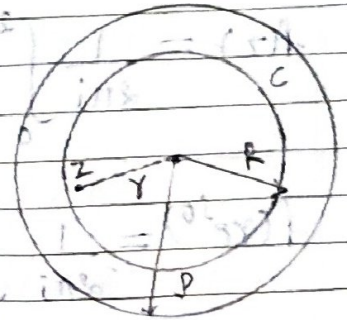
from (1) and (2)

$$f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{\bar{z}}} \right] d\zeta$$

writing $z = re^{i\theta}$ $\zeta = Re^{i\phi}$

$$\bar{z} = re^{-i\theta} \quad d\zeta = Rie^{i\phi} d\phi$$

Teacher's Signature.....



$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \left[\frac{\zeta + \frac{R^2}{\bar{z}} - \zeta + z}{(\zeta - z)(\zeta - \frac{R^2}{\bar{z}})} \right] d\zeta$$

$$f(re^{j\theta}) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(re^{j\theta} - \frac{R^2}{re^{-j\theta}}) R e^{j\phi} d\phi}{(Re^{j\phi} - re^{j\theta})(Re^{j\phi} - \frac{R^2}{re^{-j\theta}})}$$

$$f(re^{j\theta}) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(r^2 - R^2) e^{j\theta} f(Re^{j\phi}) R e^{j\phi} d\phi}{(Re^{j\phi} - re^{j\theta}) R (re^{j\phi} - Re^{-j\theta})}$$

$$f(re^{j\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{j\phi}) e^{j\theta} e^{j\phi} d\phi}{(Re^{j\phi} - re^{j\theta}) e^{j\theta} e^{j\phi} (re^{-j\theta} - Re^{-j\phi})}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{j\phi}) d\phi}{(Re^{j\phi} - re^{j\theta})(Re^{-j\phi} - re^{-j\theta})}$$

$$f(re^{j\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{j\phi}) d\phi}{R^2 - Rr(e^{j(\phi-\theta)} + e^{-j(\phi-\theta)}) + r^2}$$

$$[e^{j(\phi-\theta)} + e^{-j(\phi-\theta)} = 2\cos(\phi-\theta)]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{j\phi}) d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2}$$

Teacher's Signature.....

$$f(re^{j\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \dots$$

$$f(Re^{j\theta}) = u(r, \theta) + jv(r, \theta)$$

$$f(Re^{j\phi}) = u(R, \phi) + jv(R, \phi)$$

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) u(R, \phi) d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) v(R, \phi) d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) v(R, \phi) d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

which is Poisson's Integral formula.

Liouville's Theorem

Statement -

If $f(z)$ is an integral function and is bounded for all values of z , then it is constant.

Proof - Let z_1, z_2 be two arbitrary distinct points in the z -plane, let C be a circle with centre z_1 and radius R such that the point z_2 is interior to C . Then by Cauchy integral formula

$$f(z_2) - f(z_1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_2} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1} dz$$

$$= \frac{1}{2\pi i} \int_C f(z) \left[\frac{1}{z - z_2} - \frac{1}{z - z_1} \right] dz$$

$$= \frac{1}{2\pi i} \int_C f(z) \left[\frac{z - z_1 - z + z_2}{(z - z_2)(z - z_1)} \right] dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z) (z - z_1)}{(z - z_2)(z - z_1)} dz$$

Let us choose R so large such that

$$|z_2 - z_1| < \frac{R}{2}$$

The equation of circle C is

$$|z - z_1| = R$$

we have

$$|z - z_2| = |z - z_1 + z_1 - z_2|$$

$$= |z - z_1 - (z_2 - z_1)|$$

$$|z - z_2| \geq |z - z_1| - |z_2 - z_1|$$

$$|z - z_2| > R - \frac{R}{2}$$

$$|z - z_2| > \frac{R}{2}$$

Let M denotes the upper bound of $f(z)$ for all z lying in the finite region of the z plane. Then

$$|f(z)| \leq M$$

$$|f(z_2) - f(z_1)| = \left| \frac{z_2 - z_1}{2\pi i} \int_C \frac{f(z)}{(z - z_1)(z - z_2)} dz \right|$$

$$\leq \frac{|z_2 - z_1|}{2\pi} \int_C \frac{|f(z)|}{|z - z_1| |z - z_2|} |dz|$$

$$\leq \frac{|z_2 - z_1|}{2\pi} \frac{M}{R \times \frac{R}{2}} \int_C |dz|$$

$$\therefore \int_C |dz| = 2\pi R$$

$$= \frac{|z_2 - z_1| M 2\pi R}{2\pi \frac{R^2}{2}}$$

$$|f(z_2) - f(z_1)| \leq \frac{2|z_2 - z_1| M}{R}$$

when $R \rightarrow \infty$

Right hand side tends to 0.

$$|f(z_2) - f(z_1)| = 0$$

$$f(z_2) = f(z_1)$$

Since this holds for all couple of points z_1, z_2 in the z -plane, $f(z)$ is a constant.

Teacher's Signature.....

18/16

Date 8/1/1

P-202

Taylor's Theorem

Statement -

Let $f(z)$ be analytic at all points within a circle C_0 with centre z_0 and radius R and let z be any points C . Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)(z - z_0)^2}{2!} + \dots + \frac{f^{(n)}(z_0)(z - z_0)^n}{n!} + \dots$$

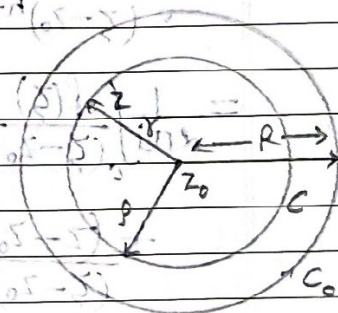
Proof - Let z be any point inside circle C_0 with centre z_0 and radius R and let

$$|z - z_0| = r$$

and let C be a circle with centre z_0 and radius ρ such that

$$r < \rho < R$$

Then point z lie inside C



By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi$$

Teacher's Signature.....

where $R_n = \frac{(z-z_0)^n}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-z)(\zeta-z_0)^n}$

Let $|z-z_0| = r$ $|z-z_0| = \rho$

$|\zeta-z| = |\zeta-z_0 + z_0-z|$
 $= |\zeta-z_0 - z-z_0|$

$|\zeta-z| \geq |\zeta-z_0| - |z-z_0|$

$|\zeta-z| \geq (\rho-r)$ and $|f(z)| \leq M$

$|R_n| = \left| \frac{(z-z_0)^n}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-z)(\zeta-z_0)^n} \right|$

$|R_n| \leq \frac{|z-z_0|^n}{2\pi} \int_C \frac{|f(\zeta)| \cdot |d\zeta|}{|\zeta-z| |\zeta-z_0|^n}$

$|R_n| \leq \frac{r^n M}{2\pi (\rho-r) \rho^n} \int_C |d\zeta|$

$\therefore \int_C |d\zeta|$ circumference of circle C
 $= 2\pi\rho$

$|R_n| \leq \frac{r^n M}{2\pi (\rho-r) \rho^n} \cdot 2\pi\rho$

$|R_n| \leq \left(\frac{r}{\rho}\right)^n \frac{M}{2\pi (\rho-r) \rho^n}$

$\therefore r < \rho$

$\Rightarrow \frac{r}{\rho} < 1$ when $n \rightarrow \infty$ $\left(\frac{r}{\rho}\right)^n \rightarrow 0$

$R_n \rightarrow 0$

$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{n!} f^n(z_0)$

which is known as Taylor's series.

Laurent's Theorem

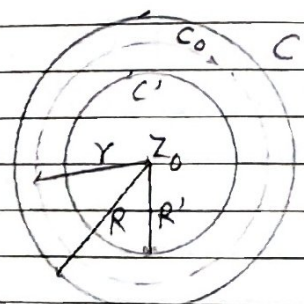
Th^m - Let $f(z)$ be analytic in the ring-shaped region D between two concentric circles C and C' with centre z_0 and radii R and R' ($R > R'$). Let z be any point of D . Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)^{n+1}}$

$b_n = \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta) d\zeta}{(\zeta-z_0)^{n-1}}$
 $n = 1, 2, 3, \dots$

Proof - Let z be any point in annulus region D between two concentric circle C and C' with centre z_0 and radii R and R' . Then by Cauchy integral formula for doubly connected region D , we have



Teacher's Signature.....

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{\zeta-z} - \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta) d\zeta}{\zeta-z}$$

$$= \frac{1}{2\pi i} \left[\int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0+z_0-z)} - \int_{C'} \frac{f(\zeta) d\zeta}{(\zeta-z_0+z_0-z)} \right]$$

$$= \frac{1}{2\pi i} \left[\int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)-(z-z_0)} - \int_{C'} \frac{f(\zeta) d\zeta}{(\zeta-z_0)-(z-z_0)} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)} \left[\frac{1-(z-z_0)}{(\zeta-z_0)} \right]^{-1} + \int_{C'}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)} \left[\frac{1-(z-z_0)}{(\zeta-z_0)} \right]$$

$$= \frac{1}{2\pi i} \left[\int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)} \left[\frac{1-(z-z_0)}{(\zeta-z_0)} \right] + \int_{C'} \frac{f(\zeta) d\zeta}{(z-z_0) \left[\frac{1-(\zeta-z_0)}{(z-z_0)} \right]} \right]$$

$$= \frac{1}{2\pi i} \left[\int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)} \left[\frac{1-(z-z_0)}{(\zeta-z_0)} \right]^{-1} + \int_{C'} \frac{f(\zeta) d\zeta}{(z-z_0)} \right]$$

$$= \frac{1}{2\pi i} \left[\int_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)} \left[1 + \frac{(z-z_0)}{(\zeta-z_0)} + \frac{(z-z_0)^2}{(\zeta-z_0)^2} + \dots + \frac{(z-z_0)^{n-1}}{(\zeta-z_0)^{n-1}} + \frac{(z-z_0)^n}{(\zeta-z_0)^n} \left[\frac{1-(z-z_0)}{(\zeta-z_0)} \right]^{-1} \right] d\zeta \right]$$

Teacher's Signature.....

$$\begin{aligned}
 & + \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)} \left[1 + \frac{(\zeta - z_0)}{(\zeta - z_0)} + \frac{(\zeta - z_0)^2}{(\zeta - z_0)^2} + \dots \right] \\
 & \left[\frac{(\zeta - z_0)^{n-1}}{(\zeta - z_0)^{n-1}} + \frac{(\zeta - z_0)^n}{(\zeta - z_0)^n} \left(1 - \frac{\zeta - z_0}{\zeta - z_0} \right)^{-1} \right] d\zeta \\
 & = \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)} d\zeta + \frac{z - z_0}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \\
 & + \frac{(z - z_0)^2}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta + \dots + \frac{(z - z_0)^{n-1}}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^n} d\zeta \\
 & + \frac{(z - z_0)^n}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \left(1 - \frac{z - z_0}{\zeta - z_0} \right) d\zeta \\
 & + \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)} d\zeta + \frac{1}{2\pi i} \int_{C'} \frac{(\zeta - z_0)}{(\zeta - z_0)^2} f(\zeta) d\zeta \\
 & + \frac{1}{2\pi i} \int_{C'} \frac{(\zeta - z_0)^2}{(\zeta - z_0)^3} f(\zeta) d\zeta + \dots + \frac{1}{2\pi i} \int_{C'} \frac{(\zeta - z_0)^{n-1}}{(\zeta - z_0)^n} f(\zeta) d\zeta \\
 & + \int_{C'} \frac{(\zeta - z_0)^n}{(\zeta - z_0)^n} \left(1 - \frac{\zeta - z_0}{\zeta - z_0} \right) f(\zeta) d\zeta \\
 & = \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)} d\zeta + \frac{z - z_0}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)} d\zeta + \dots \\
 & + \frac{(z - z_0)^n}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^n} d\zeta + R_n + \frac{1}{2\pi i} \int_{C'} \frac{1}{(\zeta - z_0)} d\zeta
 \end{aligned}$$

Teacher's Signature.....

$$\begin{aligned}
 & \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta + \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^2} (\zeta - z_0) d\zeta \\
 & + \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^3} (\zeta - z_0)^2 d\zeta + \dots + \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^n} (\zeta - z_0)^{n-1} d\zeta \\
 & + R_n' \\
 & \text{where } R_n = \frac{(z - z_0)^n}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^n (\zeta - z)} d\zeta \\
 & R_n' = \frac{1}{2\pi i} \int_{C'} \frac{(\zeta - z_0)^n}{(\zeta - z_0)^{n+1}} f(\zeta) d\zeta
 \end{aligned}$$

$$\begin{aligned}
 & \text{Writing } a_n = \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \\
 & b_n = \frac{1}{2\pi i} \int_{C'} (\zeta - z_0)^{n-1} f(\zeta) d\zeta
 \end{aligned}$$

$$\begin{aligned}
 f(z) & = [a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \\
 & + a_{n-1}(z - z_0)^{n-1} + R_n] \\
 & + \left[\frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_n}{(z - z_0)^n} + R_n' \right]
 \end{aligned}$$

Teacher's Signature.....

$$|R_n| = \left| \frac{1}{2\pi i} (z-z_0)^n \int_C \frac{f(z)}{(z-z_0)^n (z-z)} dz \right|$$

$$|R_n| \leq \frac{1}{2\pi} |z-z_0|^n \int_C \frac{|f(z)| |dz|}{|z-z_0|^n |z-z|}$$

$$\because |z-z_0| = r \quad |z-z_0| = R \quad |z-z| = |z-z_0|$$

$$+ |z-z| \geq |z-z_0| - |z-z_0|$$

$$|z-z| \geq R-r$$

$$|R_n| \leq \frac{r^n}{2\pi} \int_C \frac{M |dz|}{R^n (R-r)} = \frac{r^n M}{2\pi R^n (R-r)} \int_C |dz|$$

$$= \frac{r^n M}{2\pi R^n (R-r)} \cdot 2\pi R$$

$$|R_n| \leq \left(\frac{r}{R}\right)^n \frac{M}{1-\frac{r}{R}}$$

$$\because r < R$$

when $n \rightarrow \infty \quad \frac{r}{R} \rightarrow 0$

$$|R_n| \rightarrow 0$$

$$|R'_n| \leq \frac{1}{2\pi} |z-z_0|^{n+1} \int_C \frac{|z-z_0|^n}{|z-z|} |f(z)| |dz|$$

$$|R'_n| \leq \frac{1}{2\pi}$$

$$|z-z_0| = R'$$

$$|z-z| \geq R-R'$$

$$|R'_n| \leq \frac{1}{2\pi r^{n+1}} \int_C \frac{R'^n M'}{r-R'} |dz|$$

$$|R'_n| \leq \frac{1}{2\pi r^{n+1}} \frac{R'^n M}{r-R'}$$

$$|R'_n| \leq M \left(\frac{R'}{r}\right)^n \frac{1}{(1-\frac{R'}{r})}$$

$$\frac{R'}{r} \rightarrow 0$$

 $n \rightarrow \infty$

$$|R'_n| \rightarrow 0$$

$$f(z) = [a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + b_1 + b_2(z-z_0)^{-1} + b_3(z-z_0)^{-2} + \dots]$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

Zero of Analytic Function

A zero of an analytic function $f(z)$ is any value of z for which $f(z)$ vanishes.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{--- (1)}$$

where $a_n = \frac{f^{(n)}(a)}{n!}$

Case-I If $z=a$

$$f(a) = 0$$

and $f'(a) \neq 0$

$\Rightarrow z=a$ is simple zero of $f(z)$

Case-II If at $z=a$ $f(a) = 0 = f'(a) = \dots = f^{(m-1)}(a)$

but $f^{(m)}(a) \neq 0$

then $z=a$ is called zero of m^{th} order of $f(z)$

Pole — If there exist a positive Integer m such that

$$\lim_{z \rightarrow a} (z-a)^m f(z) = b \neq 0$$

then $z=a$ is called a pole of order m

Ex- Let $f(z) = \frac{1}{(z-1)^2 (z-3)^5}$

then $z=1$ is pole of order 2 and $z=3$ is pole of order 3.

Singular points

A singular point of function $f(z)$ is the point at which the function ceases to be analytic

Ex- $f(z) = \frac{1}{z-1}$ has singularity at $z=1$.

Isolated singularity

A point is said to be isolated singularity of function $f(z)$. If $f(z)$ is analytic

at each point in some neighbourhood

$$|z-a| < \delta$$

Ex - $f(z) = \frac{z+1}{z(z^2+2)}$

Isolated singular points are

$$z=0, z=\pm\sqrt{2}i$$

Pole - $\lim_{z \rightarrow a} (z-a)^m f(z) = b \neq 0$

then $z=a$ is called a pole of order m .

Meromorphic function

A function $f(z)$ which has pole as its only singularities in the finite part of the plane is called a meromorphic function.

P-265

19, 16, 13

Poles and Zeros of a Meromorphic Function

Theorem-11 Number of zeros and poles of a meromorphic function.

If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles and has no zero on C , then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

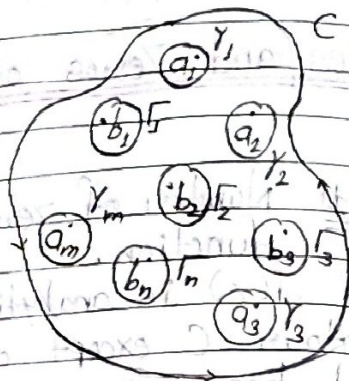
where N is the number of zeros and P the number of poles inside C .

Proof - Given that $f(z)$ is analytic in and on closed contour C and let $z = a_j$ ($j=1, 2, \dots, m$) are zeros of $f(z)$ inside C and n_j is order of a_j and b_k is number of poles ($k=1, 2, \dots, n$) of $f(z)$ inside C with p_k as a order b_k . Then we will show that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_{j=1}^m n_j - \sum_{k=1}^n p_k$$

Now we describe circle at each zero's and poles. Let $\gamma_1, \gamma_2, \dots, \gamma_m$ are circles at $a_1, a_2, \dots, a_m$ and $\Gamma_1, \Gamma_2, \dots, \Gamma_n$

are circles at $b_1, b_2, \dots, b_n$ respectively and radius of each circle is ρ .



Then by Cauchy's Goursat theorem

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_{j=1}^m \frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)} dz + \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{f'(z)}{f(z)} dz \quad (1)$$

Since a_j are zeros of n_j order of $f(z)$ so we may write

$$f(z) = (z - a_j)^{n_j} \phi_j(z)$$

where $\phi_j(z)$ is analytic and $\phi_j(a_j) \neq 0$.

Now taking logarithm of both side

$$\log f(z) = n_j \log(z - a_j) + \log \phi_j(z)$$

Differentiating, we get

$$\frac{f'(z)}{f(z)} = \frac{n_j}{z - a_j} + \frac{\phi_j'(z)}{\phi_j(z)}$$

Teacher's Signature.....

$\therefore \phi_j(z)$ and $\phi_j'(z)$ are analytic so $\frac{\phi_j'(z)}{\phi_j(z)}$ is also analytic

$$\text{Hence } \int_{\gamma_j} \frac{\phi_j'(z)}{\phi_j(z)} dz = 0 \quad (j = 1, 2, \dots, m) \quad (2)$$

$$\begin{aligned} \sum_{j=1}^m \frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)} dz &= \sum_{j=1}^m \frac{1}{2\pi i} \int_{\gamma_j} \frac{n_j}{(z - a_j)} dz \\ &- \sum_{j=1}^m \frac{1}{2\pi i} \int_{\gamma_j} \frac{\phi_j'(z)}{\phi_j(z)} dz \end{aligned}$$

using (2)

and putting $z - a_j = \rho e^{i\theta}$
 $dz = i\rho e^{i\theta} d\theta$

$$= \sum_{j=1}^m \frac{1}{2\pi i} n_j \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta$$

$$= \frac{1}{2\pi} \sum_{j=1}^m n_j \int_0^{2\pi} d\theta$$

$$= \frac{1}{2\pi} \sum_{j=1}^m n_j \int_0^{2\pi} d\theta$$

Teacher's Signature.....

$$\therefore \sum_{j=1}^m n_j = N$$

$$\sum_{j=1}^m \frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)} dz = N \quad (3)$$

Since b_k is a pole of order p_k of $f(z)$ we may write

$$f(z) = \frac{\psi_k(z)}{(z-b_k)^{p_k}}$$

again taking logarithm

$$\log f(z) = \log \psi_k(z) - p_k \log(z-b_k)$$

Differentiating

$$\frac{f'(z)}{f(z)} = \frac{\psi'_k(z)}{\psi_k(z)} - \frac{p_k}{(z-b_k)}$$

again since $\frac{\psi'_k(z)}{\psi_k(z)}$ is analytic

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{\psi'_k(z)}{\psi_k(z)} dz$$

Teacher's Signature.....

$$- \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{p_k}{(z-b_k)} dz$$

$$\therefore \frac{1}{2\pi i} \int_{\gamma_k} \frac{\psi'_k(z)}{\psi_k(z)} dz = 0$$

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{f'(z)}{f(z)} dz = - \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{p_k}{(z-b_k)} dz$$

$$= - \frac{1}{2\pi i} \sum_{k=1}^n p_k \int_0^{2\pi} \frac{\rho j e^{j\theta} d\theta}{\rho e^{j\theta}} \quad \because z-b_k = \rho e^{j\theta} \quad dz = \rho j e^{j\theta} d\theta$$

$$= - \frac{1}{2\pi} \sum_{k=1}^n p_k \cdot 2\pi$$

$$\therefore \sum_{k=1}^n p_k = P$$

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{f'(z)}{f(z)} dz = 1 - P \quad (4)$$

from (1) (3) and (4)

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

This proves the theorem.

Teacher's Signature.....

Argument Principle

Theorem-2 If $f(z)$ is analytical within and on a closed contour C having N zeros inside C but zeros on C , then

$$N = \frac{1}{2\pi} \Delta_C \arg f(z)$$

where $\Delta_C \arg f(z)$ denotes the variation in $\arg f(z)$ as z moves around C (a zero of order m being counted m times).

Proof - We know that if $f(z)$ is analytic within and on closed contour C and if total number of zeros inside C is N and poles is P then we have

$$\frac{1}{2\pi j} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

Put $f(z) = Re^{j\phi}$ so that

$$|f(z)| = R$$

$$\phi = \arg f(z) \text{ then}$$

$$f'(z) dz = df(z) = d(Re^{j\phi})$$

$$= Re^{j\phi} j d\phi + e^{j\phi} dR$$

$$f'(z) dz = df(z) = e^{j\phi} (Rj d\phi + dR)$$

$$N - P = \frac{1}{2\pi j} \int_C \frac{e^{j\phi} (Rj d\phi + dR)}{Re^{j\phi}}$$

$$= \frac{1}{2\pi j} \int_C \left(\frac{dR}{R} + j d\phi \right)$$

$$N - P = \frac{1}{2\pi j} \int_C \frac{dR}{R} + \frac{j}{2\pi j} \int_C d\phi$$

$$\text{Now } \int_C \frac{dR}{R} = [\log R]_C$$

$$\int_C \frac{dR}{R} = 0 \quad \because C \text{ is closed}$$

$$\text{also } \int_C d\phi = [\phi]_C = \Delta_C \arg f(z)$$

Since $f(z)$ does not return to its original value as z moves around C so $\Delta_C \arg f(z)$ is not necessarily zero.
therefore

$$N - P = \frac{1}{2\pi} \Delta_C \arg f(z)$$

Which is argument principle.

Teacher's Signature.....

Teacher's Signature.....

Rouche's Theorem

Theorem-(4) Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $|g(z)| < |f(z)|$ on C , then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C .

Proof - Given that $f(z)$ and $g(z)$ both are analytic in and on C and

$$|g(z)| < |f(z)| \quad \text{--- (1)}$$

$\therefore g(z)$ can not be negative on C

$$\Rightarrow f(z) > 0 \quad \because f(z) > g(z)$$

and $f(z) \neq 0$ on C

as if $f(z) + g(z) = 0$ on C

$$f(z) = -g(z)$$

$$|f(z)| = |-g(z)|$$

$$|f(z)| = |g(z)|$$

which is contradicting of (1)

So $f(z) + g(z) \neq 0$ on C

Let N and M be the number of zeros of $f(z)$ and $f(z) + g(z)$ inside C

Then by argument principle

Alternately -

$$\text{Let } F(z) = \frac{g(z)}{f(z)} \quad \text{--- (2)}$$

$$\text{Since } |g(z)| < |f(z)| \Rightarrow \frac{|g(z)|}{|f(z)|} < 1 \text{ on } C$$

$$\Rightarrow |F(z)| < 1 \text{ on } C$$

$$\text{Now } g(z) = F(z) \cdot f(z)$$

$$g' = F'f + Ff'$$

Since $f(z)$ and $f(z) + g(z)$ are analytic inside and on C we have

$$N = \frac{1}{2\pi i} \int_C \frac{f' + g'}{f + g} dz$$

$$M = \frac{1}{2\pi i} \int_C \frac{f'}{f} dz$$

$$N-M = \frac{1}{2\pi i} \int_C \frac{f'+g'}{f+g} dz - \frac{1}{2\pi i} \int_C \frac{f'}{f} dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{f'+g'}{f+g} - \frac{f'}{f} \right] dz$$

$$\therefore g = Ff$$

$$g' = F'f + Ff'$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{f' + F'f + Ff' - f'}{f + Ff} \right] dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f'f + F'f^2 + Fff' - ff' - Fff'}{f^2(1+F)}$$

$$= \frac{1}{2\pi i} \int_C \frac{F'}{1+F} dz = 0$$

$1+F \neq 0$ on C .

Since derivative of analytic function is analytic hence F' is analytic

$$\Rightarrow \frac{F'}{1+F} \text{ is analytic}$$

Hence by Cauchy-Goursat theorem

$$\frac{1}{2\pi i} \int_C \frac{F'}{1+F} dz = 0$$

$$N-M=0$$

$$N=M$$

This completes the proof of the theorem.

2) 17, 16, 14, 12, 10, 9

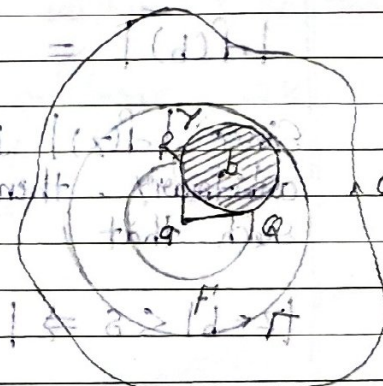
p-272

Maximum Modulus Principle

Theorem - Let $f(z)$ be analytic within and on a simple closed contour C . Then the maximum value of $|f(z)|$ occurs on C and not inside C , unless $f(z)$ is constant.

Proof - Given that $f(z)$ is analytic within and on a simple closed contour C hence continuous within and on C .

And let $f(z)$ attain its maximum value M at some point within or on C .



Now we will prove that which point where $f(z)$ attains maximum is on C not inside C .

If possible the maximum value not attained at on boundary C but it attained at a point $z = a$ within C .

$$\max |f(z)| = |f(a)| = M \quad \text{--- (1)}$$

$$\text{and } |f(z)| \leq M \quad \text{--- (2)}$$

$\forall z$ within C .

Now we describe a circle Γ with in C with centre a and $f(z)$ is not constant and let $z = b$ is any point inside Γ such that

$$|f(b)| < M \quad \text{--- (3)}$$

for any $\epsilon > 0$ let

$$|f(b)| = M - \epsilon \quad \text{--- (4)}$$

Since $|f(z)|$ is continuous at b for any arbitrary. then for given $\epsilon > 0 \exists \delta > 0$ such that

$$|z - b| < \delta \Rightarrow ||f(z)| - |f(b)|| < \frac{\epsilon}{2}$$

$\therefore$ by property of absolute value

$$|f(z)| - |f(b)| \leq ||f(z)| - |f(b)||$$

$$|f(z)| - |f(b)| \leq \frac{\epsilon}{2}$$

$$|f(z)| \leq |f(b)| + \frac{\epsilon}{2}$$

$$|f(z)| \leq M - \epsilon + \frac{\epsilon}{2} \quad \text{using (4)}$$

$$|f(z)| \leq M - \frac{\epsilon}{2} \quad \text{--- (5)}$$

$\forall z \in \gamma$ where $\gamma: |z - b| < \delta$
or γ is circle with centre b and radius

Now again describe another circle Γ' with centre a and it passes through b the arc QR of the circle Γ' lies within γ so that

$$|f(z)| \leq M - \frac{\epsilon}{2} \quad \text{on arc } QR$$

and remaining arc of Γ'

$$|f(z)| \leq M$$

Suppose r is the radius of circle Γ'

Then $|b-a| = r$

By Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(z)}{z-a} dz$$

on Γ' $z-a = re^{j\theta}$
 $dz = rie^{j\theta} d\theta$

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{j\theta}) rie^{j\theta} d\theta}{re^{j\theta}}$$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{j\theta}) d\theta$$

Let $1 \leq r \leq R = \alpha$

then

$$f(a) = \frac{1}{2\pi} \int_0^\alpha f(a+re^{j\theta}) d\theta + \frac{1}{2\pi}$$

$$\int_\alpha^{2\pi} f(a+re^{j\theta}) d\theta$$

$$|f(a)| \leq \frac{1}{2\pi} \int_0^\alpha |f(a+re^{j\theta})| d\theta + \frac{1}{2\pi}$$

$$\int_\alpha^{2\pi} |f(a+re^{j\theta})| d\theta$$

$$< \frac{1}{2\pi} \int_0^\alpha (M - \frac{1}{2}\epsilon) d\theta + \frac{1}{2\pi} \int_\alpha^{2\pi} M d\theta$$

$$= \frac{\alpha}{2\pi} (M - \frac{\epsilon}{2}) + \frac{M}{2\pi} (2\pi - \alpha)$$

$$= \frac{M\alpha}{2\pi} - \frac{\alpha\epsilon}{4\pi} + M - \frac{M\alpha}{2\pi}$$

$$|f(a)| = M - \frac{\alpha\epsilon}{4\pi}$$

but $|f(a)| = M$

$$M = M - \frac{\alpha\epsilon}{4\pi}$$

hypothesis

Which is contradiction hence our supposition is wrong. Consequently $|f(z)|$ not attain its maximum on C .

Schwarz Lemma

Theorem - If $f(z)$ is analytic in a domain $|z| < 1$ and satisfies the conditions

$$|f(z)| \leq 1, \quad f(0) = 0$$

$$\text{then } |f(z)| \leq |z| \text{ and } |f'(0)| \leq 1$$

Equality holds only if $f(z)$ is a linear transformation $w = f(z) = e^{i\alpha} z$ where α is a real constant.

Proof - Given that $f(z)$ is analytic in a domain $|z| < 1$ and satisfies the conditions

$$|f(z)| \leq 1 \quad \text{--- (1)}$$

$$f(0) = 0 \quad \text{--- (2)}$$

now by Taylor's expansion of $f(z)$

$$\text{at } z=0$$

$$f(0) = a_0 \quad \text{and} \quad f(0) = 0$$

$$0 = a_0$$

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \dots \quad \text{--- (3)}$$

$$\text{and let } F(z) = \frac{f(z)}{z}$$

$$F(z) = a_1 + a_2 z + a_3 z^2 + \dots$$

$$F(z) = a_1 + a_2 z + a_3 z^2 + \dots$$

$$\text{at } z=0$$

$$F(0) = a_1 \quad \text{--- (4)}$$

from (3)

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \dots$$

$$z=0$$

$$f'(0) = a_1 \quad \text{--- (5)}$$

from (4) and (5)

$$F(0) = f'(0) = a_1 \quad \text{--- (6)}$$

Let $z = ar$ be an arbitrary point of unit disc $|z| < 1$ let us choose r such that

$$|a| < r < 1$$

$$\text{so } |z| \leq 1$$

$$|F(z)| = \frac{|f(z)|}{|z|} \quad \text{on circle } |z| = r$$

$$|F(z)| = \frac{|f(z)|}{|z|} < \frac{1}{r}$$

$$|F(z)| < \frac{1}{r} \quad (7)$$

by maximum modulus principle $z = a$

$$|F(a)| = \frac{|f(a)|}{a} < \frac{1}{r}$$

letting $r = 1$ on boundary

$$|F(a)| = \frac{|f(a)|}{|a|} < 1$$

$$|f(a)| < |a| \quad (8)$$

$$\Rightarrow |F(a)| < 1$$

in particular $|F(0)| = |f'(0)|$

$$|f'(0)| < 1$$

from (8)

$$|f(a)| < |a| \quad \forall a \in |z| < 1$$

In general

$$|f(z)| < |z|$$

Now we conclude that $|F(a)| = 1$ can hold only when

$$F(z) = \frac{f(z)}{z} = e^{j\alpha}$$

$$f(z) = ze^{j\alpha}$$

$$f(z) = cz$$

where $c = e^{j\alpha}$

$$|c| = |e^{j\alpha}| = 1$$

This proves the assertion of the Schwarz Lemma

Residue Theory and Calculus of Residues

The Residue at a singularity

Residue of $f(z)$ at $z=a$ is given by the formula

$$b_1 = \frac{1}{2\pi i} \int_{\gamma} f(z) dz$$

Let $z=a$ is a simple pole then we have

$$b_1 = \lim_{z \rightarrow a} (z-a) f(z)$$

Calculation of Residues in some special cases

(i) $f(z) = \frac{\phi(z)}{\psi(z)}$

Residue of $f(z)$ at $z=a$ where $\phi(z) \neq 0$ and $\psi(z)$ has a simple pole at $z=a$

$$\text{Residue}_{z=a} f(z) = \lim_{z \rightarrow a} \frac{(z-a) \phi(z)}{\psi(z)}$$

(ii) When $f(z)$ has a pole of order M at $z=a$

$$\text{Residue}_{z=a} f(z) = b_1 = \frac{\phi^{(m-1)}(a)}{(m-1)!}$$

Example-(3) Evaluate the residues of $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at the poles $z=1, 2, 3$

Solution-

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$$

here $z=1$ is a pole of order 4

Putting $z-1 = t$

Now, $\text{Res}_{z=1} f(z) = \text{Coefficient of } \frac{1}{t} \text{ in } \frac{1}{t^4}$

$$\left[\frac{(1+t)^3}{(t-1)(t-2)} \right]$$

$$= \text{Coefficient of } \frac{1}{t} \text{ in } \frac{1}{t^4} \left[(1+t)^3 (1-t)^{-1} \left(1 - \frac{1}{2}t\right)^{-1} \right]$$

$$= \text{Coefficient of } \frac{1}{t} \text{ in } \frac{1}{t^4} \left[(1+3t+3t^2+t^3) \right]$$

$$(1+t+t^2+t^3+t^4+t^5+\dots)$$

$$\left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \frac{t^5}{32} + \dots \right)$$

= Coefficient of $\frac{1}{t}$ in

$$\begin{aligned} & \frac{1}{2+4} \left[(1+3+3^2+3^3) \left(1 + \frac{3}{2} + \frac{7}{4} + \frac{15}{8} \right) \right] \\ &= \frac{1}{2} \times \left(\frac{15}{8} + \frac{21}{4} + \frac{9}{2} + 1 \right) \\ &= \frac{1}{2} \times \frac{(15+42+36+8)}{8} \\ &= \frac{101}{16} \end{aligned}$$

Ans

$$\text{Res}_{z=1} f(z) = \frac{101}{16}$$

$$\begin{aligned} \text{Res}_{z=2} f(z) &= \lim_{z \rightarrow 2} (z-2) \\ f(z) &= \lim_{z \rightarrow 2} \frac{z^3}{(z-1)^4(z-3)} = -8 \end{aligned}$$

$$\begin{aligned} \text{Res}_{z=3} f(z) &= \lim_{z \rightarrow 3} (z-3) \\ f(z) &= \lim_{z \rightarrow 3} \frac{z^3}{(z-1)^4(z-2)} = \frac{27}{16} \end{aligned}$$

14.12.10

P-301

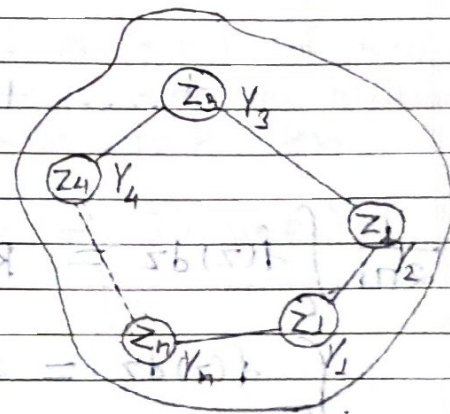
Cauchy's Residue Theorem

Theorem - If $f(z)$ is regular, except at a finite of poles $z_1, z_2, \dots, z_n$ within a closed contour C where its residues are $R_1, R_2, \dots, R_n$, respectively and continuous on the boundary C , then

$$\oint_C f(z) dz = 2\pi i (R_1 + R_2 + \dots + R_n)$$

Proof - Given that $f(z)$ is analytic in contour C and let $z_1, z_2, \dots, z_n$ are points inside C now we describe circle $\gamma_1, \gamma_2, \dots, \gamma_n$ with centre $z_1, z_2, \dots, z_n$ respectively and radius of each circle ρ .

And circle are not overlap each other. They by Cauchy's Theorem form multiply connected region. We have



$$\oint_C f(z) dz = \sum_{i=1}^n \int_{\gamma_i} f(z) dz = 0$$

$$\int_C f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \dots + \int_{\gamma_n} f(z) dz \quad (1)$$

by definition of Residue we know that

$$\frac{1}{2\pi i} \int_{\gamma_1} f(z) dz = \text{Res}_{z=z_1} f(z) = R_1$$

Hence from (1)

$$\frac{1}{2\pi i} \int_C f(z) dz = \frac{1}{2\pi i} \int_{\gamma_1} f(z) dz + \frac{1}{2\pi i} \int_{\gamma_2} f(z) dz + \dots + \frac{1}{2\pi i} \int_{\gamma_n} f(z) dz$$

$$\frac{1}{2\pi i} \int_C f(z) dz = R_1 + R_2 + \dots + R_n$$

$$\int_C f(z) dz = 2\pi i \sum R$$

($2\pi i$ (sum of residue at $z_1, z_2, \dots$)

This completes the proof of the theorem.

P-303

Some Important Statement

(1) If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z-a| = r$ and if $\lim_{z \rightarrow a} (z-a) f(z) = A$, then

$$\lim_{r \rightarrow 0} \int_C f(z) dz = iA(\theta_2 - \theta_1)$$

(2) If C is an arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and if $\lim_{R \rightarrow \infty} z f(z) = A$, then

$$\lim_{R \rightarrow \infty} \int_C f(z) dz = i(\theta_2 - \theta_1)A$$

or

If AB is an arc of the circle $|z| = R$ having $\theta_1 \leq \theta \leq \theta_2$ and when R tends to infinity, $z f(z)$ tends uniformly to k , then

$$\lim_{R \rightarrow \infty} \int_{AB} f(z) dz = i(\theta_2 - \theta_1)k$$

Example (1) Show that

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a > b > 0$$

Solution - Given Integral $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

Let C be a unit circle $|z| = 1$ and

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, \quad a > b > 0$$

Putting $z = e^{j\theta}$ so that $dz = je^{j\theta} d\theta$

$$\therefore \cos\theta = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad \frac{dz}{je^{j\theta}} = d\theta$$

$$\frac{dz}{jz} = d\theta$$

$$I = \frac{1}{j} \int_C \frac{dz}{z \left[a + \frac{b}{2} \left(z + \frac{1}{z} \right) \right]}$$

$$= \frac{1}{j} \int_C \frac{2dz}{z [2az + bz^2 + b]}$$

$$I = \frac{2}{j} \int_C \frac{dz}{bz^2 + 2az + b}$$

$$\text{Let } f(z) = \frac{1}{bz^2 + 2az + b}$$

$$I = \frac{2}{j} \int_C f(z) dz \quad \text{--- (1)}$$

Pole of $f(z)$ are given by roots of $bz^2 + 2az + b = 0$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$\text{Let } \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}, \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

$$\text{Here } |\alpha\beta| = |\alpha||\beta| = 1$$

Here $z = \alpha$ is simple pole inside C and β is out side from unit circle

$$|z| = 1$$

$$f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

$$\text{Res}_{z=\alpha} f(z) = \lim_{z \rightarrow \alpha} (z-\alpha) f(z)$$

$$= \lim_{z \rightarrow \alpha} (z-\alpha) \frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \lim_{z \rightarrow \alpha} \frac{1}{b(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{b} \frac{1}{-a + \sqrt{a^2 - b^2} - (-a - \sqrt{a^2 - b^2})}$$

$$= \frac{1}{b} \frac{2\sqrt{a^2 - b^2}}{b}$$

$$\text{Res}_{z=\alpha} f(z) = \frac{2\sqrt{a^2 - b^2}}{2\sqrt{a^2 - b^2}}$$

$$\text{Hence } \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{j} \int_C f(z) dz = \frac{2\pi j \times 2}{2j \sqrt{a^2 - b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Proved

Teacher's Signature.....

Example (3)(i) Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ ($a > b > 0$).

Solution - Putting $z = e^{j\theta}$ $\frac{dz}{jz} = d\theta$

$$\cos\theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

$$I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{j} \int_C \frac{dz}{z \left(a + \frac{b}{2} \left(z + \frac{1}{z} \right) \right)^2}$$

$$= \frac{1}{j} \int_C \frac{dz}{(2az + bz^2 + b)^2}$$

$$= -4j \int_C \frac{z dz}{(bz^2 + 2az + b)^2}$$

$$= -\frac{4j}{b^2} \int_C \frac{z dz}{(z^2 + 2\frac{a}{b}z + 1)^2}$$

$$\text{Let } \frac{z}{z^2 + 2\frac{a}{b}z + 1} = f(z)$$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = -\frac{4j}{b^2} \int_C f(z) dz \quad \text{--- (1)}$$

Teacher's Signature.....

where C is unit circle $|z| = 1$
 Poles of order two of $f(z)$ are given by
 the roots of

$$z^2 + \frac{2a}{b}z + 1 = 0$$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\left(\frac{4a^2}{b^2} - 4\right)}}{2}$$

$$z = \frac{-\frac{2a}{b} \pm 2\sqrt{a^2 - b^2}}{2} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$\text{let } \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Since $|\beta| \leq 1$ and $\alpha\beta = 1$, $|\alpha| < 1$

Hence only α lies inside $|z| = 1$

$$\text{Now } \frac{-4i}{b^2} f(z) = \frac{-4iz}{b^2(z-\alpha)^2(z-\beta)^2}$$

$$= \frac{\phi(z)}{(z-\alpha)^2}$$

$$\text{where } \phi(z) = \frac{-4iz}{b^2(z-\beta)^2}$$

Residue at the double pole $z = \alpha$ is

$$\text{Res}_{z=\alpha} f(z) = \frac{\phi'(\alpha)}{1!}$$

$$\phi'(z) = -\frac{4i}{b^2} \left[\frac{(z-\beta)^2 - 2z(z-\beta)}{(z-\beta)^4} \right]$$

$$= -\frac{4i}{b^2} \left[\frac{(z-\beta) - 2z}{(z-\beta)^3} \right]$$

$$= \frac{4i}{b^2} \left[\frac{z + \beta}{(z-\beta)^3} \right]$$

$$\phi'(\alpha) = \frac{4i}{b^2} \left[\frac{\alpha + \beta}{(\alpha - \beta)^3} \right]$$

$$\therefore \alpha + \beta = -\frac{2a}{b}$$

$$\alpha - \beta = \frac{2\sqrt{a^2 - b^2}}{b}$$

$$\phi'(\alpha) = \frac{4i}{b^2} \left[\frac{-\frac{2a}{b}}{\left(\frac{2\sqrt{a^2 - b^2}}{b}\right)^3} \right]$$

$$= 4i \left[\frac{-2a}{8(a^2-b^2)^{3/2}} \right]$$

$$\phi(z) = \frac{-ia}{(a^2-b^2)^{3/2}}$$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = 2\pi i \left[\frac{-ai}{(a^2-b^2)^{3/2}} \right]$$

$$= \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Ans.

P-311 2) 157

Example-(7) Show that

$$\int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} = \frac{\pi a^2}{1-a^2} \quad (a^2 < 1)$$

Solution—Let $I = \int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2}$

$$I = \int_0^\pi \frac{\cos 2\theta \{(1+a^2)+2a\cos\theta\} d\theta}{\{(1+a^2)-2a\cos\theta\} \{(1+a^2)-2a\cos\theta\}}$$

$$I = \int_0^\pi \frac{(1+a^2)\cos 2\theta d\theta}{(1+a^2)^2-4a^2\cos^2\theta} + 2a$$

$$\int_0^\pi \frac{\cos 2\theta \cos \theta d\theta}{(1+a^2)^2-4a^2\cos^2\theta}$$

$$\therefore \int_0^\pi \frac{\cos 2\theta \cos \theta d\theta}{(1+a^2)^2-4a^2\cos^2\theta} = 0$$

$$I = \int_0^\pi \frac{(1+a^2)\cos 2\theta d\theta}{(1+a^2)^2-2a^2(1+\cos 2\theta)}$$

by property of definite integrals

Putting $2\theta = \phi \quad d\theta = \frac{d\phi}{2}$

$$I = \frac{1}{2} \int_0^{2\pi} \frac{(1+a^2)\cos \phi d\phi}{(1+a^2)^2-2a^2(1+\cos \phi)}$$

Putting $\cos \phi = \frac{1}{2}(z + \frac{1}{z})$

$d\phi = \frac{dz}{jz}$

$$I = \frac{1}{2} \int_C \frac{(1+a^2) \frac{1}{2}(z + \frac{1}{z}) \cdot \frac{dz}{jz}}{(1+a^2)^2-2a^2\left[1 + \frac{1}{2}\left(z + \frac{1}{z}\right)\right]}$$

$$\begin{aligned}
 &= \frac{1}{4i} \int_C \frac{(1+a^2)(z^2+1)}{z(1+a^2)^2 - \frac{2a^2}{z}(z^2+2z+1)} dz \\
 &= \frac{1}{4i} (1+a^2) \int_C \frac{(z^2+1)}{z(1+a^2)^2 - a^2(z^2+2z+1)} dz \\
 &= \frac{1}{4i} (1+a^2) \int_C \frac{(z^2+1) dz}{z\{a^2 z^2 + 2a^2 z + a^2 - (a^4 + 2a^2 z + 1)\}} \\
 &= \frac{(1+a^2)}{4i} \int_C \frac{(z^2+1) dz}{z\{a^2 z^2 - (a^4 + 1)z + a^2\}} \\
 &= \frac{(1+a^2)}{4ia^2} \int_C \frac{z^2+1}{z\{z^2 - (a^2 + \frac{1}{a^2})z + a^2\}} dz \\
 &= \frac{(1+a^2)}{4ia^2} \int_C \frac{z^2+1}{z(z-a^2)(z-\frac{1}{a^2})} dz \\
 f(z) &= \frac{z^2+1}{z(z-a^2)(z-\frac{1}{a^2})}
 \end{aligned}$$

$f(z)$ has simple pole at $z=0$, $z=a^2$
and $z=\frac{1}{a^2}$ $\because a < 1$

$z=0$ and $z=a^2$ lie inside C

$$\begin{aligned}
 \text{Res}_{z=0} f(z) + \text{Res}_{z=a^2} f(z) &= \lim_{z \rightarrow 0} z f(z) + \lim_{z \rightarrow a^2} (z-a^2) f(z) \\
 &= \frac{(1+a^2)i}{4a^2} \left\{ \lim_{z \rightarrow 0} \frac{z(z^2+1)}{z(z-a^2)(z-\frac{1}{a^2})} + \lim_{z \rightarrow a^2} \frac{(z-a^2)(z^2+1)}{z(z-a^2)(z-\frac{1}{a^2})} \right\} \\
 &= \frac{(1+a^2)i}{4a^2} \left\{ \frac{1}{a^2(a^2-1)} + \frac{a^2+1}{a^2(a^2-1)} \right\} \\
 &= \frac{(1+a^2)i}{4a^2} \left[\frac{2a^2}{a^2(a^2-1)} \right] = \frac{a^2 i}{2(a^2-1)}
 \end{aligned}$$

Hence $\int_0^\pi \frac{\cos 2\theta + d\theta}{1-2a \cos \theta + a^2} = \frac{2\pi i \cdot a^2 i}{2(a^2-1)}$

$$= \frac{\pi a^2}{1-a^2}$$

Evaluation of the Integral of the Type

$$\int_{-\infty}^{\infty} f(x) dx$$

P-321 25

Example-(1) By the method of Contour integration prove that

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

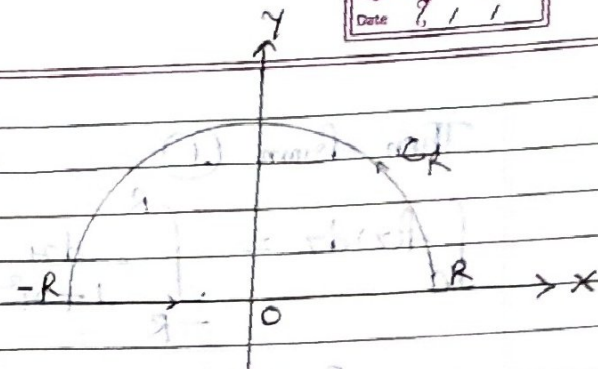
Solution- Given Integral

$$\int_0^{\infty} \frac{dx}{1+x^2} = \int_C f(z) dz$$

where C is contour consisting of a large semi-circle C_R of radius R together with the part of the real axis from $-R$ to R then by residue theorem

$$\int_C f(z) dz = \int_{-R}^R \frac{dx}{1+x^2} + \int_{C_R} \frac{dz}{1+z^2} = 2\pi i \sum R \quad (1)$$

Teacher's Signature.....



$$\lim_{|z| \rightarrow \infty} z f(z) = \lim_{|z| \rightarrow \infty} \frac{z}{1+z^2}$$

$$= \lim_{z \rightarrow \infty} \frac{1}{\frac{1}{z} + z}$$

taking limit

$$= \frac{1}{\frac{1}{\infty} + \infty} = \frac{1}{0 + \infty} = 0$$

$$\lim_{|z| \rightarrow \infty} z f(z) = 0$$

They by theorem

$$\Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} \frac{dz}{1+z^2} = 0 \quad (2)$$

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Teacher's Signature.....

Then from (1)

$$\int_C f(z) dz = \int_{-R}^R \frac{dx}{1+x^2} = 2\pi j \sum R^+$$

$R \rightarrow \infty$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi j \sum R^+$$

Now $f(z)$ has simple poles at

$z = \pm j$ of which $z = j$ lies inside C

$$\text{Res}_{z=j} f(z) = \lim_{z \rightarrow j} (z-j) \frac{1}{1+z^2}$$

$$= \lim_{z \rightarrow j} (z-j) \frac{1}{(z-j)(z+j)}$$

$$\text{Res}_{z=j} f(z) = \frac{1}{2j}$$

Hence from (2)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2 \int_0^{\infty} \frac{dx}{1+x^2} = 2\pi j \cdot \frac{1}{2j}$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} \quad \text{Ans}$$

Example 2(c) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Solution - Consider integral $\int_C \frac{dz}{(1+z^2)^3} = \int_C f(z) dz$

$$f(z) = \frac{1}{(1+z^2)^3}$$

Where C is contour consisting of a large semi circle C_R of radius R together with the part of real axis from $x = -R$ to $x = R$ then by residue theorem.

$$\int_C f(z) dz = \int_{-R}^R \frac{dx}{(1+x^2)^3} + \int_{C_R} \frac{dz}{(1+z^2)^3}$$

$$= 2\pi j \sum R \quad (1)$$

Since

$$\lim_{|z| \rightarrow \infty} z f(z) = \lim_{z \rightarrow \infty} z \frac{1}{1+z^2}$$

$$= \lim_{z \rightarrow \infty} \frac{1}{z} \frac{1}{1+z}$$

$$= \frac{1}{\infty + \infty}$$

$$= \frac{1}{\infty}$$

$$\lim_{|z| \rightarrow \infty} z f(z) = 0$$

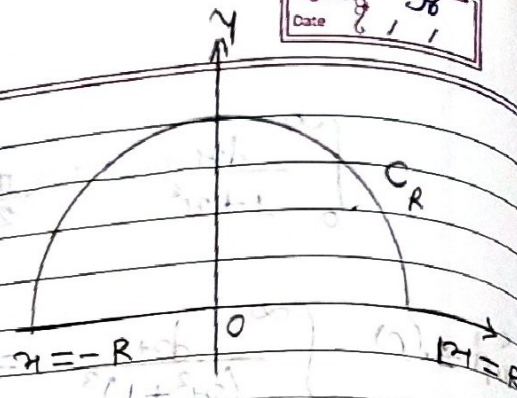
$$\Rightarrow \lim_{R \rightarrow \infty} \int_C \frac{dz}{(1+z^2)^3} = 0$$

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{(1+x^2)^3} + \lim_{R \rightarrow \infty} \int_C \frac{dz}{(1+z^2)^3} dz = 2\pi i \sum R$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = 2\pi i \sum R \quad (2)$$

$f(z)$ has poles order 3 at $z = \pm i$. $z = i$ lies inside upper half plane.
Hence Residue at $z = i$

$$\text{Res}_{z=i} f(z) = \frac{1}{2!} \phi''(i) \text{ where } \phi(z) = \frac{1}{(z+i)^3}$$



$$\phi'(z) = \frac{-3}{(z+i)^4}$$

$$\phi''(z) = \frac{-3 \times -4}{(z+i)^5}$$

$$\phi''(z) = \frac{12}{(z+i)^5}$$

$$z = i$$

$$\phi''(i) = \frac{12}{(i+i)^5} = \frac{12}{(2i)^5}$$

$$\Rightarrow \frac{\phi''(i)}{2!} = \frac{12^2}{32i \times 2} = \frac{3}{8i \times 2}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx = 2\pi i \frac{3}{16i}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx = \frac{3\pi}{8}$$

Proved

Jordan's Lemma

If (i) $f(z) \rightarrow 0$ as $|z| \rightarrow \infty$ uniformly for $\arg z \leq \pi$, and

(ii) $f(z)$ is meromorphic in the upper half plane,

then $\lim_{R \rightarrow \infty} \int_{C_R} e^{imz} f(z) dz = 0 \quad (m > 0)$

where C_R denotes the semi-circle $|z| = R$, $\Im(z) \geq 0$

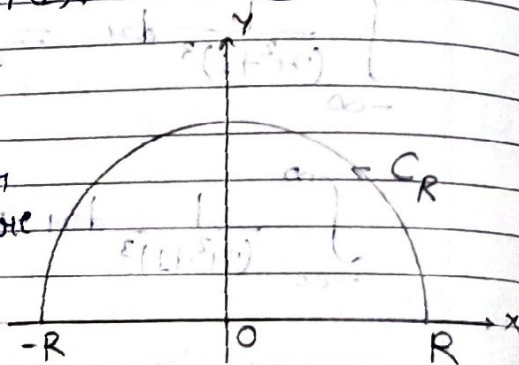
P-336

Ex-1) Apply the method of residues to prove that

Proof - Consider a integral: $\int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$

$$\int_C \frac{e^{miz}}{a^2 + z^2} dz = \int_C F(z) dz \quad \text{--- (1)}$$

where C is contour as shown in figure then by Cauchy integral formula



where $F(z) = \frac{e^{miz}}{a^2 + z^2}$

$$\int_C F(z) dz = \int_{-R}^R e^{imx} f(x) dx + \int_{C_R} e^{imz} f(z) dz = 2\pi i \sum R \quad \text{--- (2)}$$

where $f(z) = \frac{1}{a^2 + z^2}$

taking $R \rightarrow \infty$ in (2)

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2 + x^2} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{imz}}{a^2 + z^2} dz = 2\pi i \sum R \quad \text{--- (3)}$$

$$\text{Since } f(z) = \lim_{z \rightarrow \infty} \frac{1}{a^2 + z^2} = \frac{1}{\infty} = 0$$

Hence by Jordan's Lemma

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{imz} f(z) dz = \lim_{R \rightarrow \infty} \int_{C_R} e^{imz} \frac{1}{z^2 + a^2} dz = 0$$

from (3)

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2 + x^2} dx = 2\pi i \sum R \quad \text{--- (4)}$$

Pole of $f(z)$ are $z = +ai$ and $z = -ai$
but only $z = ai$ is lie in upper half plane.

$$\begin{aligned} \text{Res}_{z=ia} f(z) &= \lim_{z \rightarrow ai} (z - ai) f(z) \\ &= \lim_{z \rightarrow ai} (z - ai) \frac{1}{z^2 + a^2} e^{imz} \\ &= \lim_{z \rightarrow ai} (z - ai) \frac{1}{(z + ai)(z - ai)} e^{imz} \\ &= \frac{e^{im(ai)}}{2ai} \\ &= \frac{e^{-ma}}{2ai} \end{aligned}$$

from (4)

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2 + x^2} dx = \frac{2\pi i}{2ai} e^{-ma}$$

$$\int_0^{\infty} \frac{e^{imx}}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$$

$$\int_0^{\infty} \frac{\cos mx + i \sin mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$$

Equating real and imaginary part

$$\int_0^{\infty} \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$$

Poles on the Real Axis

P-361 17

Example-(9) Apply the calculus of residues to prove that

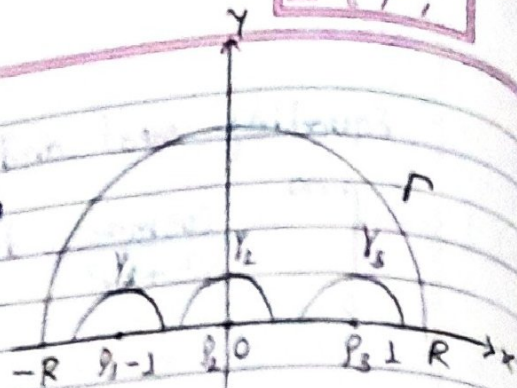
$$\int_0^{\infty} \frac{\sin \pi x}{x(1+x^2)} dx = \pi$$

Solution- Consider

$$\int_C \frac{e^{iz}}{z(1-z^2)} dz = \int_C f(z) dz \quad \text{--- (1)}$$

Where C is closed contour consisting of Γ , upper half of large circle $|z| = R$ and real axis $|z| = R$ and real from $-R$ to R and circles $\gamma_1, \gamma_2, \gamma_3$ at $z = -1, 0, 1$ respectively

Then by Cauchy integral formula



$$\begin{aligned} \int_C F(z) dz &= \int_{\gamma_1} e^{j\pi z} f(z) dz + \int_{-R}^{1-p_1} F(z) dz \\ &+ \int_{\gamma_2} F(z) dz + \int_{-1+p_1}^{0-p_2} F(z) dz + \int_{\gamma_3} F(z) dz + \int_{0+p_2}^{1-p_3} F(z) dz \\ &+ \int_{\gamma_4} F(z) dz + \int_{0+p_3}^R F(z) dz = 0 \quad \text{--- (1)} \end{aligned}$$

Taking limit $R \rightarrow \infty$ $p_1, p_2, p_3 \rightarrow 0$

$$\lim_{R \rightarrow \infty} \int_{\gamma_1} e^{j\pi z} f(z) dz + \int_{-\infty}^1 F(z) dz + \lim_{p_1 \rightarrow 0} \int_{\gamma_1} F(z) dz$$

$$+ \int_{-1}^0 F(z) dz + \lim_{p_2 \rightarrow 0} \int_{\gamma_2} F(z) dz + \int_0^1 F(z) dz +$$

$$\lim_{p_3 \rightarrow 0} \int_{\gamma_3} F(z) dz + \int_1^{\infty} F(z) dz = 0 \quad \text{--- (2)}$$

Teacher's Signature.....

$$\begin{aligned} \lim_{z \rightarrow \infty} f(z) &= \lim_{z \rightarrow \infty} \frac{1}{z(1-z^2)} \\ &= \frac{1}{\infty} = 0 \end{aligned}$$

Then by Jordan lemma

$$\lim_{R \rightarrow \infty} \int_{\gamma} e^{j\pi z} f(z) dz = 0 \quad \text{--- (4)}$$

$$\text{Here } \lim_{z \rightarrow -1} (z+1)F(z) = \lim_{z \rightarrow -1} \frac{(z+1)e^{j\pi z}}{z(1-z^2)}$$

$$= \lim_{z \rightarrow -1} \frac{(z+1)e^{j\pi z}}{z(z+1)(1-z)}$$

$$= \frac{e^{-j\pi}}{-1(-2)}$$

$$= \frac{e^{-j\pi}}{2}$$

$$= \frac{\cos \pi + j \sin \pi}{2} = \frac{1}{2}$$

Then by theorem (1) using

$$\begin{aligned} \lim_{p_1 \rightarrow 0} \int_{\gamma_1} F(z) dz &= j(0-\pi) \times \frac{1}{2} \\ &= \frac{-\pi j}{2} \end{aligned}$$

Teacher's Signature.....

Similarly $\lim_{z \rightarrow 0} (z-0) F(z) = \lim_{z \rightarrow 0} \frac{z e^{j\pi z}}{z(1-z^2)}$
 $= \frac{e^0}{1} = 1$

$\lim_{R_2 \rightarrow 0} \int_{\gamma_2} F(z) dz = j(0-\pi) \cdot 1$
 $= -\pi j$

$\lim_{z \rightarrow 1} (z-1) F(z) = \lim_{z \rightarrow 1} \frac{(z-1) e^{j\pi z}}{(1-z^2)z}$

$= \lim_{z \rightarrow 1} \frac{e^{j\pi z}}{(1+z)z}$
 $= \frac{e^{j\pi}}{2}$

$= \frac{\cos \pi + j \sin \pi}{2}$

$= \frac{1}{2}$

$\lim_{R_2 \rightarrow 0} \int_{\gamma_2} F(z) dz = -j(\pi-0) \cdot \frac{1}{2}$
 $= -\frac{\pi j}{2}$

From (3)

$0 + \int_{-\infty}^{-1} F(x) dx + \int_{-1}^0 F(x) dx + \int_0^1 F(x) dx$

$+ \int_1^{\infty} F(x) dx$

$-\frac{\pi j}{2} - \pi j - \frac{\pi j}{2} = 0$

$\int_{-\infty}^{\infty} F(x) dx = 2\pi j$

$\int_{-\infty}^{\infty} \frac{e^{j\pi x}}{x(1-x^2)} dx = 2\pi j$

Equating imaginary parts we get $\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = \pi$

$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = \pi$

$\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = \frac{\pi}{2}$

Integrals Many Valued Functions

P-365

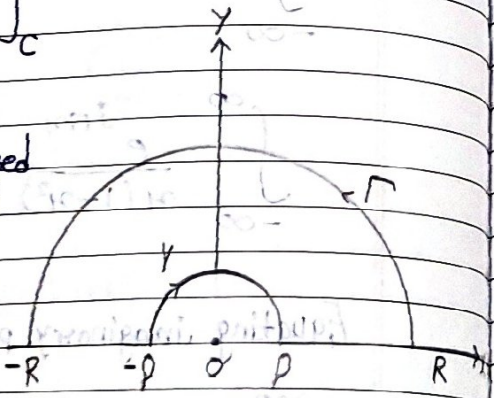
Example (2) Prove that $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx = -\frac{\pi}{4}$

using as a contour a large semicircle in the upper half plane indented at the origin.

Solution - Consider a integral

$$\int_C \frac{\log z}{(1+z^2)^2} dz = \int_C f(z) dz$$

where C is the closed contour consisting of Γ , the upper half of the large semi circle $|z|=R$ and real axis from $x=-R$



to $x=R$ and small circle γ of radius r then by Cauchy residue theorem

$$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz + \int_{-R}^{-r} f(x) dx$$

Teacher's Signature.....

$$+ \int_{\gamma} f(z) dz = 2\pi i \sum R \quad \text{--- (1)}$$

$$\begin{aligned} \lim_{z \rightarrow \infty} z F(z) &= \lim_{z \rightarrow \infty} z \frac{\log z}{(1+z^2)^2} \\ &= \lim_{z \rightarrow \infty} \frac{z^2}{(1+z^2)^2} \times \lim_{z \rightarrow \infty} \frac{\log z}{z} \\ &= \lim_{z \rightarrow \infty} \frac{1}{\left(\frac{1}{z^2} + 1\right)} \times \lim_{z \rightarrow \infty} \frac{1}{z} \\ &= \frac{1}{0+1} \times \frac{1}{\infty} \\ &= 1 \times 0 \end{aligned}$$

$$\lim_{z \rightarrow \infty} z F(z) = 0$$

Then by theorem we know that

$$\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = i(\pi - 0) \cdot 0 = 0 \quad \text{--- (2)}$$

$$\text{Again } \lim_{z \rightarrow 0} z F(z) = \lim_{z \rightarrow 0} z \frac{\log z}{(1+z^2)^2}$$

Teacher's Signature.....

$$= \lim_{t \rightarrow \infty} \frac{\frac{1}{t} \log \frac{1}{t}}{\left(1 + \frac{1}{t^2}\right)^2}$$

$$= \lim_{t \rightarrow \infty} \frac{\frac{1}{t} \log \frac{1}{t}}{(t^2 + 1)^2} \times t^4$$

$$= \lim_{t \rightarrow \infty} \frac{t^3 \times (\log 1 - \log t)}{(1 + t^2)^2}$$

$$= \lim_{t \rightarrow \infty} \frac{t^2}{(1 + t^2)^2} \times \lim_{t \rightarrow \infty} \left(\frac{-\log t}{t} \right)$$

$$= \lim_{t \rightarrow \infty} \frac{1}{\left(\frac{1}{t^2} + 1\right)^2} \times \lim_{t \rightarrow \infty} \frac{-\frac{1}{t}}{1}$$

$$= \frac{1}{\left(\frac{1}{\infty} + 1\right)^2} \times \frac{-1}{\infty}$$

$$= -1 \times 0$$

$$\lim_{z \rightarrow 0} z F(z) = 0$$

Then

$$\Rightarrow \lim_{t \rightarrow 0} \int_{\gamma} f(z) dz = i(0 - \pi) \cdot 0 = 0$$

New pole of $F(z)$ is given by $(1 + z^2)^2 = 0$

$z = \pm i \Rightarrow +i$ is lie inside of C

$$\text{Res}_{z=i} F(z) = \frac{\phi(i)}{1!}$$

where $\phi(z) = \frac{\log z}{(1 + z)^2}$

so that $f(z) = \frac{\phi(z)}{(z - i)^2}$

$$\phi'(z) = \frac{\frac{1}{z} (z + i)^2 - \log z \cdot 2(z + i)}{(z + i)^4}$$

$z = i$

$$\phi'(z) = \frac{1}{1} \frac{(i + i) - 2i \log i}{(2i)^3}$$

$$= \frac{2i - 2 \log i \times i}{8i^3}$$

$$= \frac{i - i \log i}{4}$$

$$= \frac{i(1 - \log i)}{4}$$

$$= \frac{i}{4} \left[1 - \log e^{i\pi/2} \right]$$

$$\phi'(i) = \frac{i}{4} \left[i + \frac{\pi}{2} \right]$$

$$\text{Residue } F(z) \text{ at } z=i = \frac{\phi'(i)}{1!} = \frac{1}{4} \left[\frac{\pi}{2} + i \right]$$

Taking $R \rightarrow \infty$, $\rho \rightarrow 0$ in (1)

$$\int_0^\infty F(x) dx + \lim_{R \rightarrow \infty} \int_\gamma F(z) dz + \int_{-\infty}^0 F(z) dz$$

$$\lim_{R \rightarrow \infty} \int_\gamma F(z) dz = 2\pi i \times \frac{1}{4} \left[\frac{\pi}{2} + i \right]$$

$$= \int_{-\infty}^\infty F(x) dx = \frac{2\pi i}{4} \left[\frac{\pi}{2} + i \right]$$

$$2 \int_0^\infty \frac{\log x}{(1+x^2)^2} dx = \frac{2\pi i}{4} \left[\frac{\pi}{2} + i \right]$$

$$2 \int_0^\infty \frac{\log x}{(1+x^2)^2} dx = \frac{\pi i}{2} \left(\frac{\pi}{2} + i \right)$$

$$2 \int_0^\infty \frac{\log x}{(1+x^2)^2} dx = \frac{\pi}{2} \left(\frac{\pi i}{2} - 1 \right)$$

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx = \frac{\pi}{4} \left(-1 + \frac{\pi i}{2} \right)$$

Equating to real part from both side

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx = \frac{\pi}{4}$$

Bilinear TransformationsBilinear or Mobius TransformationDefinition — The transformation T defined by

$$w = T(z) = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

where a, b, c, d are complex constant and $ad-bc \neq 0$.
 w and z are complex variable.

Inverse transformation

If $w = T(z) = \frac{az+b}{cz+d}$ is mobius

transformation then the inverse transformation of T is defined by

$$z = T^{-1}(w) = \frac{-dw+b}{cw-a} \quad \text{--- (2)}$$

P-428

Example — Consider the transformations

$$w = T_1(z) = \frac{z+1}{z+3}, \quad w = T_2(z) = \frac{z}{z+2}$$

and find $T_1^{-1}(w)$, $T_2^{-1}(w)$, $T_2 T_1(z)$, $T_1 T_2(z)$, $T_1 T_2^{-1}(w)$ and $T_2^{-1} T_1(z)$.

Solution — Given transformation

$$w = T_1(z) = \frac{z+1}{z+3} \quad \text{--- (1)}$$

$$w = T_2(z) = \frac{z}{z+2} \quad \text{--- (2)}$$

$$\frac{z+1}{z+3} = w \Rightarrow \frac{z+1-w}{z+3} = w-1$$

$$\frac{z+1-w}{z+3} = w-1$$

$$\frac{-2}{z+3} = w-1$$

$$z+3 = \frac{-2}{w-1}$$

$$z = \frac{-2}{w-1} - 3$$

$$Z = \frac{-2-3W+3}{W-1}$$

$$Z = \frac{1-3W}{W-1}$$

$$Z = T_1^{-1}(W) = \frac{-3W+1}{W-1} \quad \text{--- (3)}$$

Again

$$W = T_2(Z) = \frac{Z}{Z+2}$$

$$W = \frac{Z}{Z+2}$$

$$\frac{Z+2}{Z} = \frac{1}{W}$$

$$1 + \frac{2}{Z} = \frac{1}{W}$$

$$\frac{2}{Z} = \frac{1}{W} - 1$$

$$\frac{2}{Z} = \frac{1-W}{W}$$

$$Z = \frac{2W}{1-W}$$

Teacher's Signature.....

$$Z = T_2^{-1}(W) = \frac{2W}{1-W}$$

$$Z = T_2^{-1}(W) = \frac{-2W}{W-1} \quad \text{--- (4)}$$

$$T_1 T_2(Z) = T_1 T_2(Z) \quad \therefore \text{from (2)}$$

$$= T_1\left(\frac{Z}{Z+2}\right)$$

$$= \frac{\frac{Z}{Z+2} + 1}{\frac{Z}{Z+2} + 3} \quad \therefore \text{(1) using}$$

$$= \frac{Z + Z + 2}{Z + 3Z + 6}$$

$$= \frac{Z + Z + 2}{Z + 3Z + 6}$$

$$= \frac{2Z + 2}{4Z + 6}$$

$$= \frac{2(Z+1)}{2(2Z+3)}$$

$$T_1 T_2(Z) = \frac{Z+1}{2Z+3}$$

Teacher's Signature.....

$$T_1 T_2^{-1}(w) = T_1 [T_2^{-1}(w)] \quad \text{using (7)}$$

$$= T_1 \left[\frac{-2w}{w-1} \right] \quad \text{using (1)}$$

$$= \frac{-2w}{w-1} + 1$$

$$= \frac{-2w + w - 1}{w-1}$$

$$= \frac{-w-1}{w-1}$$

$$T_2^{-1} T_1(z) = T_2^{-1} \left(\frac{z+1}{z+3} \right)$$

$$= -2 \left(\frac{z+1}{z+3} \right)$$

$$= \frac{-2(z+1)}{z+3}$$

$$= \frac{-2(z+1)}{-2}$$

$$= z+1$$

932

Cross Ratio

Definition — If z_1, z_2, z_3, z_4 are distinct points taken in this order then the cross ratio of these points is defined as

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

Theorem — 1. Cross Ratio are invariant under a bilinear transformation.

Proof — Let w_1, w_2, w_3, w_4 be the images of the four distinct points z_1, z_2, z_3, z_4 in the z -plane under a bilinear transformation.

$$w = T(z) = \frac{az+b}{cz+d} \quad (ad-bc) \neq 0$$

Now we have

$$w_1 = \frac{az_1+b}{cz_1+d} \quad w_2 = \frac{az_2+b}{cz_2+d}$$

$$w_3 = \frac{az_3 + b}{cz_3 + d} \quad w_4 = \frac{az_4 + b}{cz_4 + d}$$

$$\begin{aligned} w_1 - w_2 &= \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d} \\ &= \frac{ac/z_1 z_2 + adz_1 - bc z_2 + bd}{(cz_1 + d)(cz_2 + d)} \\ &= \frac{(ad - bc)z_1 - (ad - bc)z_2}{(cz_1 + d)(cz_2 + d)} \\ &= \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \end{aligned}$$

Similarly

$$w_3 - w_4 = \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)}$$

$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

$$w_4 - w_1 = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}$$

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}$$

$$= \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \times \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)} \div \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \times \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}$$

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

or

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

p-435

Example-1. Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ into the point $w_1 = 1$, $w_2 = i$ and $w_3 = -1$.

Solution- We know that by invariant of cross ratio

$$\frac{(w - w_1)(w_2 - w_3)}{(w_3 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_3 - z_2)(z_3 - z)}$$

$$\frac{(w - 1)(i + 1)}{(1 - i)(-1 - w)} = \frac{(z - 2)(i + 2)}{(2 - i)(-2 - z)}$$

$$\frac{-(w-1)(j+1)(j+1)}{(w+1)(1-j)(1+j)} = \frac{(7-2)(j+2)(j+2)}{(7+2)(2+j)(2-j)}$$

$$\frac{-(w-1)(-1+2j+1)}{(w+1)(1-j)} = \frac{+(7-2)(-1+4j+4)}{(7+2)(4+1)}$$

$$\frac{(w-1)2j}{(w+1)2} = \frac{(7-2)(4j+3)}{(7+2)5}$$

$$\frac{w-1}{w+1} = \frac{(7-2)(4-3j)}{(7+2)5}$$

$$\frac{w-1-1}{w+1} = \frac{(7-2)(4-3j)-1}{(7+2)5}$$

$$\frac{w-1-2-1}{w+1} = \frac{47-3-3j7+6j-57-10}{5(7+2)}$$

$$\frac{-2}{w+1} = \frac{-7-3j(7-2)-18}{5(7+2)}$$

$$\frac{w+1}{-1} = \frac{5(7+2)}{-7-3j(7-2)-18}$$

$$w = \frac{10(7+2)}{7+3j(7-2)+18}$$

$$w = \frac{10+20-7-3j(7-2)-18}{7+3j(7-2)+18}$$

$$w = \frac{(12-7)-3j(7-2)}{(7+18)+3j(7-2)}$$

$$= \frac{-7(3j+1)+6(j+2)}{7(2j+1)+6(-j+3)}$$

$$= \frac{-7+6(j+2)}{(3j+1)}$$

$$\frac{7+6(j+3)}{(3j+1)}$$

$$= \frac{-7+6(j+2)(1-3j)}{1-9}$$

$$\frac{7+6(-j+3)(1-3j)}{1-9}$$

$$= \frac{-7-\frac{3}{4}(j+3+2-5j)}{7-\frac{3}{4}(-j-2+3-9j)}$$

$$\frac{-7-\frac{3}{4}(-5j+5)}{7-\frac{3}{4}(-10j)}$$

$$= \frac{-7-\frac{3}{4}(-5j+5)}{7-\frac{3}{4}(-10j)}$$

$$= 47-3$$

p-451

Example-1 Find the fixed points and the normal form of the following bilinear transformation

(ii) $W = \frac{z-1}{z+1}$ — (1)

Solution— Putting $W=z$ in (1)

$$z = \frac{z-1}{z+1}$$

$$z^2 + z = z - 1$$

$$z^2 + 1 = 0$$

$$(z+i)(z-i) = 0$$

$$z = i, -i$$

Thus i and $-i$ are two fixed points in this case

Now we will find Normal form.

$$W-i = \frac{z-1}{z+1} - i$$

$$= \frac{z-1-i(z+1)}{z+1}$$

$$W-i = \frac{(1-i)z - (1+i)}{z+1} \quad (vi)$$

$$W+i = \frac{z-1}{z+1} + i$$

$$= \frac{z-1+i(z+1)}{z+1}$$

$$= \frac{(1+i)z + (i-1)}{z+1}$$

$$\frac{W-i}{W+i} = \frac{(1-i)z - (1+i)}{z+1} \times \frac{z+1}{(1+i)z + (i-1)}$$

$$= \frac{(1-i)z - (1+i)}{(1+i)z + (i-1)}$$

$$= \frac{(1-i)z - (1-i)i}{(1+i)z + (1+i)i}$$

$$= \frac{(1-i)(z-i)}{(1+i)(z+i)} \cdot \frac{(1-i)}{(1-i)}$$

$$= \frac{(1-i)^2(z-i)}{(1+i)(z+i)}$$

$$\frac{W-i}{W+i} = -i \frac{(z-i)}{(z+i)}$$

Here $\lambda = -1$
Teacher's Signature.....

$$(iv) W = \frac{3iz+1}{z+i}$$

Solution - putting $w = z$

$$z = \frac{3iz+1}{z+i}$$

$$z^2 + iz = 3iz + 1$$

$$z^2 + iz - 3iz - 1 = 0$$

$$z^2 - 2iz - 1 = 0$$

$$(z-i)^2 = 0$$

$$z = i, i$$

This $z=i$ is only fixed point. Consequently the transformation is parabolic.

To put the transformation in the normal form we have

$$W = \frac{3iz+1}{z+i}$$

$$Wz + wi - 3iz - 1 = 0$$

$$(w-i)(z-i) + wi + zi + 1 + wi - 3iz - 1 = 0$$

$$(w-i)(z-i) + 2wi - 2zi = 0$$

$$(w-i)(z-i) + 2wi + 2 - 2zi - 2 = 0$$

$$\frac{(w-i)(z-i)}{(w-i)(z-i)} + \frac{2i(w-i)}{(w-i)(z-i)} - \frac{2i(z-i)}{(w-i)(z-i)} = 0$$

$$1 + \frac{2i}{z-i} - \frac{2i}{w-i} = 0$$

$$\frac{2i}{w-i} = \frac{2i}{z-i} + 1$$

$$\boxed{\frac{1}{w-i} = \frac{1}{z-i} + \frac{1}{2i}}$$

$$\frac{1}{w-i} = \frac{1}{z-i} + \frac{1}{2i}$$

Special Bilinear Transformations

$$W = \frac{az+b}{cz+d}$$

Theorem (I) To find all the bilinear transformation which maps the half plane $I(z) \geq 0$ onto the unit circular disc $|w| \leq 1$.

Proof - let

$$W = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

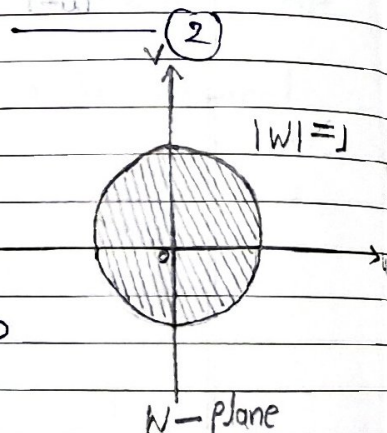
is the bilinear transformation which maps half plane $I(z) \geq 0$ onto the unit circle $|w| \leq 1$

$$W = \frac{a(z + \frac{b}{a})}{c(z + \frac{d}{c})}$$

$$\text{at } z = -\frac{b}{a} \Rightarrow W = 0$$

and

$$\text{at } z = -\frac{d}{c} \Rightarrow W = \infty$$



And we know that $w = \infty$ is inverse point of

$w = 0$ with respect to the unit circle $|w| = 1$

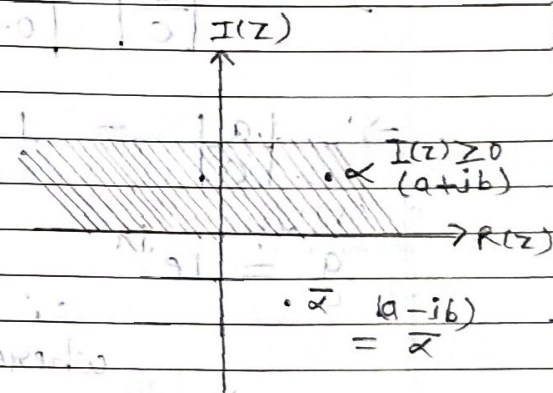
Correspond to the point $z = -\frac{b}{a}$ and

$$z = -\frac{d}{c}$$

Let $-\frac{b}{a} = \alpha$ then

Inverse point of α with respect to

line $I(z) = 0$ is $\bar{\alpha}$



$$\text{Hence } -\frac{d}{c} = \bar{\alpha}$$

Z-plane

Hence (2) reduces

$$W = \frac{a}{c} \frac{(z - \alpha)}{(z - \bar{\alpha})} \quad \text{--- (3)}$$

Determination of $\frac{a}{c}$

Since real axis $I(z) = 0$ is transformed into unit circle $|w| = 1$ at the point $z = 0$ So

$$|W| = 1$$

$$1 = |W| = \left| \frac{a}{c} \frac{(z-\alpha)}{(z-\bar{\alpha})} \right|$$

$$1 = \left| \frac{a}{c} \right| \left| \frac{0-\alpha}{0-\bar{\alpha}} \right|$$

$$\Rightarrow \left| \frac{a}{c} \right| = 1$$

$$\frac{a}{c} = 1e^{j\lambda}$$

$\therefore |\alpha| = |\bar{\alpha}|$
where λ is real

from (3)

$$W = e^{j\lambda} \frac{z-\alpha}{z-\bar{\alpha}} \quad \text{--- (4)}$$

Verification

$$\begin{aligned} \Rightarrow W\bar{W} - 1 &= e^{j\lambda} \frac{z-\alpha}{z-\bar{\alpha}} \cdot e^{-j\lambda} \frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha} - 1 \\ &= \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\bar{\alpha})(\bar{z}-\alpha)} - 1 \end{aligned}$$

$$= \frac{z\bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha} - z\bar{z} + z\alpha + \bar{z}\bar{\alpha} - \alpha\bar{\alpha}}{|z-\bar{\alpha}|^2}$$

$$= \frac{z(\alpha-\bar{\alpha}) - \bar{z}(\alpha-\bar{\alpha})}{|z-\bar{\alpha}|^2}$$

$$W\bar{W} - 1 = \frac{(z-\bar{z})(\alpha-\bar{\alpha})}{|z-\bar{\alpha}|^2}$$

$$\begin{aligned} \because z-\bar{z} &= x+jy-x+jy \\ &= 2jy \\ &= 2I(z)j \end{aligned}$$

and

$$\alpha-\bar{\alpha} = 2jI(\alpha)$$

$$W\bar{W} - 1 = \frac{2jI(z) \cdot 2jI(\alpha)}{|z-\bar{\alpha}|^2}$$

$$W\bar{W} - 1 = \frac{-4I(z)I(\alpha)}{|z-\bar{\alpha}|^2}$$

Since $I(\alpha) > 0$ and $I(z) > 0$ and $|z-\bar{\alpha}|^2 > 0$

$$W\bar{W} - 1 = \frac{-4I(z)I(\alpha)}{|z-\bar{\alpha}|^2} \leq 0$$

$$w\bar{w} - 1 < 0$$

$$|w|^2 < 1$$

$$\Rightarrow |w| < 1$$

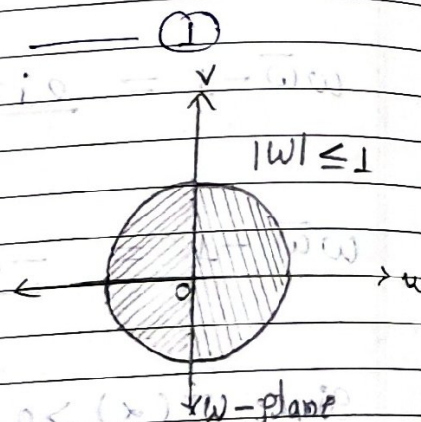
Hence $T(z) > 0$ is transformed onto $|w| < 1$.

Theorem-II Find the bilinear transformation which transforms half plane $R(z) > 0$ onto the unit circular disc $|w| < 1$.

Proof - let

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

is the transformation which maps upper half plane $R(z) > 0$ onto unit circular disc $|w| < 1$.



(1) may be written as

$$w = \frac{a(z - \frac{b}{a})}{c(z + \frac{d}{c})} \quad \text{--- (2)}$$

Since $w=0$ is inverse point of $w=\infty$ with respect to unit circle $|w|=1$ and

$$\text{at } z = -\frac{b}{a} \Rightarrow w=0$$

and

$$\text{at } z = -\frac{d}{c} \Rightarrow w=\infty$$

Then $z = -\frac{b}{a}$ should be inverse point of

$z = -\frac{d}{c}$ with respect to plane $R(z) > 0$.

$$-\bar{z} = (-a+ib)$$

$$\bar{\alpha} = (-a+ib), \quad \alpha = (a+ib)$$

so that if $-\frac{b}{a} = \alpha$

$$\Rightarrow -\frac{d}{c} = -\bar{\alpha}$$

$$w = \frac{a}{c} \frac{(z - \alpha)}{(z + \bar{\alpha})} \quad \text{--- (3)}$$

Now determination to $\frac{a}{c}$

Teacher's Signature.....

Again line $R(z)=0$ is mapped by
Circle $|w|=1$ at $z=0$

$$1=|w| \Rightarrow 1 = \left| \frac{a(z-\alpha)}{c(z+\bar{\alpha})} \right|$$

$$1 = \left| \frac{a}{c} \right| \frac{|z-\alpha|}{|z+\bar{\alpha}|}$$

at $z=0$

$$1 = \left| \frac{a}{c} \right| \frac{|0-\alpha|}{|0+\bar{\alpha}|}$$

$$1 = \left| \frac{a}{c} \right| \frac{|\alpha|}{|\bar{\alpha}|}$$

$$\therefore |\alpha| = |\bar{\alpha}|$$

$$1 = \left| \frac{a}{c} \right|$$

Hence we may write $\frac{a}{c} = 1 \cdot e^{j\lambda}$
where λ is real

$$W = e^{j\lambda} \frac{(z-\alpha)}{(z+\bar{\alpha})} \quad \text{--- (4)}$$

Verification -

$$W\bar{W} - 1 = e^{j\lambda} \frac{(z-\alpha)}{(z+\bar{\alpha})} e^{-j\lambda} \frac{(\bar{z}-\bar{\alpha})}{(\bar{z}+\alpha)} - 1$$

Teacher's Signature.....

$$= \frac{z\bar{z} - \bar{\alpha}z - \alpha\bar{z} + \alpha\bar{\alpha} - \bar{z}\bar{z} - \alpha\bar{z} - \alpha\bar{\alpha} - \bar{\alpha}\bar{z}}{|z+\bar{\alpha}|^2}$$

$$= \frac{-\bar{\alpha}(z+\bar{z}) - \alpha(z+\bar{z})}{|z+\bar{\alpha}|^2}$$

$$= \frac{-(\alpha+\bar{\alpha})(z+\bar{z})}{|z+\bar{\alpha}|^2}$$

$$\therefore \alpha+\bar{\alpha} = 2R(\alpha)$$

$$z+\bar{z} = 2R(z)$$

$$W\bar{W} - 1 = \frac{-2R(\alpha) \cdot 2R(z)}{|z+\bar{\alpha}|^2}$$

$$\therefore R(\alpha) > 0, R(z) > 0$$

$$|z+\bar{\alpha}|^2 > 0$$

$$W\bar{W} - 1 = \frac{-4R(z)R(\alpha)}{|z+\bar{\alpha}|^2} \leq 0$$

$$W\bar{W} - 1 \leq 0$$

$$|W|^2 \leq 1$$

$$|W| \leq 1$$

Hence transformation (4) is transformation $R(z)$
 > 0 to $|W| \leq 1$

Teacher's Signature.....

Example (a) 17

Theorem-III) Find all the Möbius transformation which transform the unit circle $|z| \leq 1$ onto the unit circular disc $|w| \leq 1$.

Proof - Let

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

be the required transformation

$$w = \frac{a(z + \frac{b}{a})}{c(z + \frac{d}{c})} \quad \text{--- (2)}$$

$$\text{at } z = -\frac{b}{a} \Rightarrow w = 0$$

and

$$\text{at } z = -\frac{d}{c} \Rightarrow w = \infty$$

$w = \infty$ is inverse point of $w = 0$ with respect to the unit circle $|w| = 1$. Correspond to the point $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$

Let $-\frac{b}{a} = \alpha$ with respect to the circle

$$|z| = 1 \text{ is } \frac{1}{\bar{\alpha}} \text{ so}$$

$$\text{hence } -\frac{b}{a} = \alpha, \quad -\frac{d}{c} = \frac{1}{\bar{\alpha}}$$

Hence (2) reduce

$$w = \frac{a}{c} \left[\frac{z - \alpha}{z - \frac{1}{\bar{\alpha}}} \right] = \frac{a\bar{\alpha}}{c} \left[\frac{z - \alpha}{z\bar{\alpha} - 1} \right] \quad \text{--- (3)}$$

Determination of $\frac{a}{c}$ - To find $\frac{a}{c}$ we may use the fact that any point on $|w| = 1$ must correspond to a point on $|z| = 1$. Hence putting $z = 1$

$$|w| = 1$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| \left| \frac{z - \alpha}{z\bar{\alpha} - 1} \right| = 1$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| \left| \frac{1 - \alpha}{\bar{\alpha} - 1} \right| = 1$$

$$[\because (1 - \alpha) = (\bar{\alpha} - 1)]$$

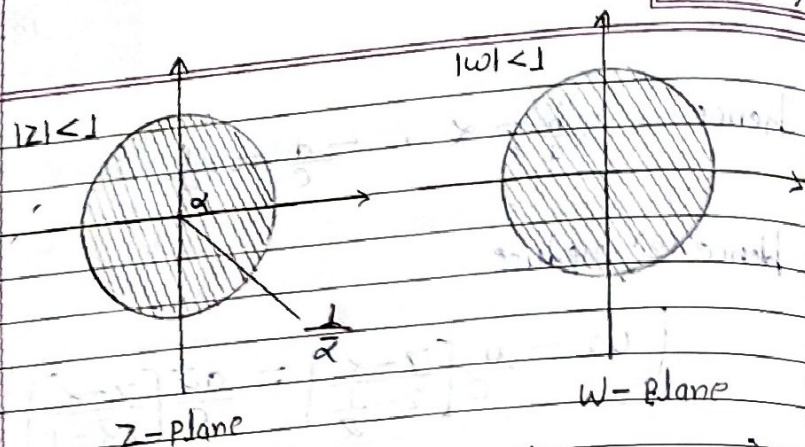
$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| = 1$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| = 1 \cdot e^{j\lambda}$$

where λ is real

from (3)

$$w = e^{j\lambda} \left[\frac{z - \alpha}{z\bar{\alpha} - 1} \right] \quad \text{--- (4)}$$



Verification — $w\bar{w}-1 = \frac{e^{j\lambda}(z-\alpha)}{(z\bar{\alpha}-1)} \frac{e^{-j\lambda}(\bar{z}-\bar{\alpha})}{(\bar{z}\alpha-1)}$

$$\Rightarrow |w|^2 - 1 = \frac{(z-\alpha)(\bar{z}-\bar{\alpha}) - (z\bar{\alpha}-1)(\bar{z}\alpha-1)}{(z\bar{\alpha}-1)(\bar{z}\alpha-1)}$$

$$\Rightarrow |w|^2 - 1 = \frac{z\bar{z} - z\bar{\alpha} - \bar{z}\alpha + \alpha\bar{\alpha} - z\bar{z}\alpha\bar{\alpha} + z\bar{\alpha} + \bar{z}\alpha - 1}{|z\bar{\alpha}-1|^2}$$

$$\Rightarrow |w|^2 - 1 = \frac{z\bar{z}(1-\alpha\bar{\alpha}) - 1}{|z\bar{\alpha}-1|^2}$$

$$= \frac{(z\bar{z}-1)(1-\alpha\bar{\alpha})}{|z\bar{\alpha}-1|^2}$$

$$\Rightarrow |w|^2 - 1 = \frac{(|z|^2-1)(1-|\alpha|^2)}{|z\bar{\alpha}-1|^2} \quad \text{--- (5)}$$

$$\because |z| < 1, |\alpha| < 1 \text{ and } |w|^2 - 1 < 0$$

$$\Rightarrow |w|^2 \leq 1 \Rightarrow |w| \leq 1$$

Hence eqn (5) shows that the transformation $|w| \leq 1$ onto $|z| \leq 1$ provided $|\alpha| < 1$.

Theorem-(IV.) Find the general bilinear transformation which maps the circular disc $|z| \leq p$ onto the circular disc $|w| \leq p'$ and show that it can be put in the form

$$w = pp' e^{j\lambda} \frac{z-\alpha}{\bar{\alpha}z-p^2} \quad (|\alpha| < p)$$

Proof —

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

be the required transformation

$$w = \frac{a}{c} \left[\frac{z + (\frac{b}{a})}{z + (\frac{d}{c})} \right] \quad \text{--- (2)}$$

$$\text{at } z = -\frac{b}{a} \Rightarrow w = 0$$

$$\text{and at } z = -\frac{d}{c} \Rightarrow w = \infty$$

$w = \infty$ is inverse point of $w = 0$ with respect to the unit circle $|w| = 1$. Correspond

to the $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$ in the z -plane

Let $-\frac{b}{a} = \alpha$ with respect to the circle

$$|z| = p' \text{ is } \frac{p^2}{\bar{z}}$$

Hence $\boxed{-\frac{b}{a} = \alpha, -\frac{d}{c} = \frac{p^2}{\bar{\alpha}}}$

Hence (2) reduce

$$\boxed{w = \frac{a}{c} \left[\frac{z-\alpha}{z-\frac{p^2}{\bar{\alpha}}} \right] = \frac{a\bar{\alpha}}{c} \left[\frac{z-\alpha}{z\bar{z}-p^2} \right]} \quad \text{--- (3)}$$

Determination of $\frac{a}{c}$ - Clearly (3) satisfies the condition $|z| \leq p$ and $|w| \leq p'$

Hence for $|z| = p$; i.e. $z\bar{z} = p^2$ we must have $|w| = p'$. Thus from (3) we have

$$|w| = p'$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| \left| \frac{z-\alpha}{z\bar{z}-p^2} \right| = p'$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| \left| \frac{z-\alpha}{z\bar{z}-p^2} \right| = p' \quad [\because z\bar{z} = p^2]$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| \frac{1}{|z|} \left| \frac{z-\alpha}{\bar{\alpha}-\bar{z}} \right| = p' \quad [\because z=p]$$

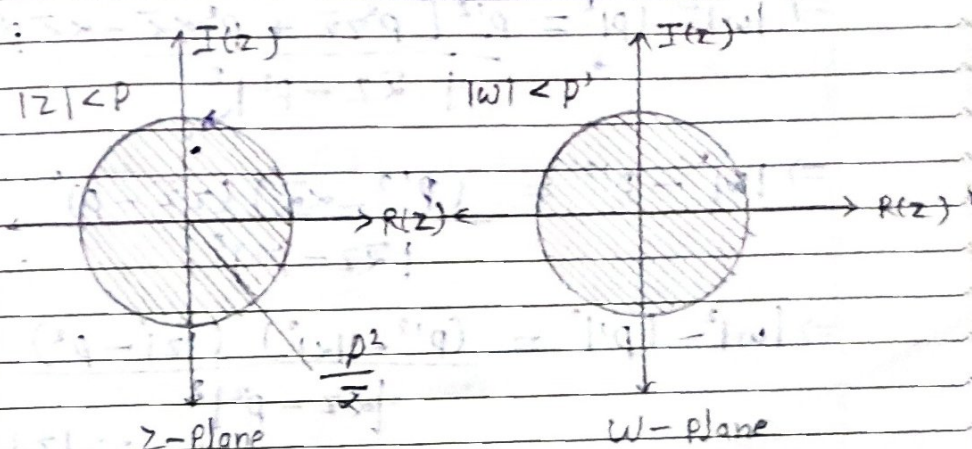
$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| \frac{1}{p} = p' \quad [|z-\alpha| = |\bar{z}-\bar{\alpha}| = |\bar{z}-\bar{\alpha}|]$$

$$\Rightarrow \left| \frac{a\bar{\alpha}}{c} \right| = pp'$$

$$\Rightarrow \frac{a\bar{\alpha}}{c} = pp' \cdot e^{j\lambda} \quad \text{where } e^{j\lambda} \text{ is real}$$

from (3)

$$\boxed{w = pp'e^{j\lambda} \left[\frac{z-\alpha}{z\bar{z}-p^2} \right]} \quad \text{--- (4)}$$



Verification -

$$w\bar{w} - p^2 = \frac{pp'e^{i\lambda}(z-\alpha)}{(z\bar{\alpha}-p^2)} \cdot pp'e^{-i\lambda}$$

$$\frac{(\bar{z}-\bar{\alpha})}{(z\bar{\alpha}-p^2)} = p'^2$$

$$\Rightarrow |w|^2 = p'^2 = \frac{p'p'^2(z-\alpha)(\bar{z}-\bar{\alpha}) - (z\bar{\alpha}-p^2)}{(z\bar{\alpha}-p^2)(\bar{z}\alpha-p^2)}$$

$$\Rightarrow |w|^2 - p'^2 = \frac{p'^2 [p^2 z\bar{z} - p^2 z\bar{\alpha} - p^2 \bar{z}\alpha + p^2 \bar{\alpha}\alpha - \alpha\bar{\alpha} \cdot z\bar{z} + p^2 \bar{z}\alpha + p^2 \alpha\bar{z} - p^4]}{(z\bar{\alpha}-p^2)(\bar{z}\alpha-p^2)}$$

$$\Rightarrow |w|^2 - |p|^2 = \frac{p'^2 [p^2 z\bar{z} + p^2 \alpha\bar{\alpha} - \alpha\bar{\alpha} p^2 - p^4]}{|z\bar{z} - p^2|^2}$$

$$\Rightarrow |w|^2 - p'^2 = \frac{(p'^2 - \alpha\bar{\alpha})(z\bar{z} - p^2)}{|z\bar{z} - p^2|^2}$$

$$\Rightarrow |w|^2 - |p|^2 = \frac{(p'^2 - |k|^2)(|z|^2 - p^2)}{|z\bar{z} - p^2|^2}$$

$$\therefore |z| < p$$

$$|k| < p$$

$$|w|^2 - p'^2 < 0$$

Teacher's Signature.....

$$\therefore 1 < |k| < p \text{ then } p^2 < |k|^2$$

$$\text{or } p^2 - |k|^2 > 0$$

Hence by the transformation (4) the circular disc $|z| \leq p$ is mapped onto the circular disc $|w| \leq p'$.

F-466

Example-(3) Show that the transformation $w = \frac{2z+3}{z-4}$

maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

Solution - The given transformation is

$$w = \frac{2z+3}{z-4} \quad \text{--- (1)}$$

$$w - 2 = \frac{2z+3-2z+8}{z-4}$$

$$w - 2 = \frac{11}{z-4}$$

$$z - 4 = \frac{11}{w-2}$$

Teacher's Signature.....

Verification -

$$w\bar{w} - p^2 = pp'e^{i\theta} \frac{(z-\alpha)}{(z\bar{\alpha}-p^2)} \cdot pp'e^{-i\theta}$$

$$\frac{(\bar{z}-\bar{\alpha})}{(\bar{z}\alpha-p^2)} = p'^2$$

$$\Rightarrow |w|^2 = p'^2 = \frac{p'p'^2(z-\alpha)(\bar{z}-\bar{\alpha}) - (z\bar{\alpha}-p^2)}{(z\bar{\alpha}-p^2)(\bar{z}\alpha-p^2)}$$

$$\Rightarrow |w|^2 - p'^2 = \frac{p'^2 [p^2 z\bar{z} - p^2 z\bar{\alpha} - p^2 \bar{z}\alpha + p^2 \bar{z}\bar{\alpha} - \alpha\bar{\alpha} \cdot z\bar{z} + p^2 \alpha\bar{\alpha} + p^2 \bar{\alpha}\alpha - p^4]}{(z\bar{\alpha}-p^2)(\bar{z}\alpha-p^2)}$$

$$\Rightarrow |w|^2 - |p|^2 = \frac{p'^2 [p^2 z\bar{z} + p^2 \alpha\bar{\alpha} - \alpha\bar{\alpha} p^2 - p^4]}{|z\bar{\alpha}-p^2|^2}$$

$$\Rightarrow |w|^2 - p'^2 = \frac{(p'^2 - |\alpha|^2)(z\bar{z} - p^2)}{|z\bar{\alpha}-p^2|^2}$$

$$\Rightarrow |w|^2 - |p|^2 = \frac{(p'^2 - |\alpha|^2)(|z|^2 - p^2)}{|z\bar{\alpha}-p^2|^2}$$

$$\therefore |z| < p$$

$$|\alpha| < p$$

$$|w|^2 - p'^2 < 0$$

Teacher's Signature.....

$$\therefore |z| < p \text{ then } p^2 < |z|^2$$

$$\text{or } p^2 - |z|^2 > 0$$

Hence by the transformation (2) the circular disc $|z| \leq p$ is mapped onto the circular disc $|w| \leq p'$.

P-466

Example - (3) Show that the transformation $w = \frac{2z+3}{z-4}$

maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

Solution - The given transformation is

$$w = \frac{2z+3}{z-4} \quad \text{--- (1)}$$

$$w - 2 = \frac{2z+3 - 2z+8}{z-4}$$

$$w - 2 = \frac{11}{z-4}$$

$$z - 4 = \frac{11}{w-2}$$

Teacher's Signature.....

Verification:-

$$w\bar{w} - p^2 = \frac{pp'e^{i\lambda}(z-\alpha)}{(z\bar{\alpha}-p^2)} \cdot \frac{pp'e^{-i\lambda}}{(z\bar{\alpha}-p^2)}$$

$$\frac{(\bar{z}-\bar{\alpha})}{(z\bar{\alpha}-p^2)} = p'^2$$

$$\Rightarrow |w|^2 = p'^2 = \frac{p'p'^2(z-\alpha)(\bar{z}-\bar{\alpha}) - (z\bar{\alpha}-p^2)}{(z\bar{\alpha}-p^2)(\bar{z}\alpha-p^2)}$$

$$\Rightarrow |w|^2 - p'^2 = \frac{p'^2 [p^2 z\bar{z} - p^2 z\bar{\alpha} - p^2 \bar{z}\alpha + p^2 \bar{z}\bar{\alpha} - \alpha\bar{\alpha} \cdot z\bar{z} + p^2 \alpha\bar{\alpha} + p^2 \bar{\alpha}\alpha - p^4]}{(z\bar{\alpha}-p^2)(\bar{z}\alpha-p^2)}$$

$$\Rightarrow |w|^2 - |p|^2 = \frac{p'^2 [p^2 z\bar{z} + p^2 \alpha\bar{\alpha} - \alpha\bar{\alpha} p^2 \bar{z} - p^4]}{|z\bar{z} - p^2|^2}$$

$$\Rightarrow |w|^2 - p'^2 = \frac{(p'^2 - \alpha\bar{\alpha})(z\bar{z} - p^2)}{|z\bar{z} - p^2|^2}$$

$$\Rightarrow |w|^2 - |p|^2 = \frac{(p'^2 - |z|^2)(|z|^2 - p^2)}{|z\bar{z} - p^2|^2}$$

$$\therefore |z| < p$$

$$|z| < p$$

$$|w|^2 - p'^2 < 0$$

$$\Rightarrow |w| < p'$$

Teacher's Signature.....

$$\therefore |z| < p \text{ then } p^2 < |z|^2$$

$$\text{or } p^2 - |z|^2 > 0$$

Hence by the transformation (2) the circular disc $|z| \leq p$ is mapped onto the circular disc $|w| \leq p'$.

P-466

Example-(3) Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

Solution— The given transformation is

$$w = \frac{2z+3}{z-4} \quad \text{--- (1)}$$

$$w - 2 = \frac{2z+3 - 2(z-4)}{z-4} = \frac{2z+3-2z+8}{z-4} = \frac{11}{z-4}$$

$$w - 2 = \frac{11}{z-4}$$

$$z - 4 = \frac{11}{w-2}$$

Teacher's Signature.....

$$z = \frac{11}{w-2} + 4$$

$$z = \frac{11 + 4w - 8}{w-2}$$

Inverse transformation is

$$z = \frac{4w+3}{w-2} \quad \text{--- (2)}$$

given equation of circle

$$x^2 + y^2 - 4x = 0$$

$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2$$

$$z\bar{z} = x^2 + y^2$$

$$z\bar{z} - 2(z+\bar{z}) = 0 \quad \text{--- (3)}$$

Now substituting z and $\bar{z}$ from (2) in (3)

$$\Rightarrow \frac{4w+3}{w-2} \cdot \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$\Rightarrow 16w\bar{w} + 12w + 12\bar{w} + 9 - 8w\bar{w} - 6\bar{w} + 16w + 12 - 8w\bar{w} - 6w + 16\bar{w} + 12 = 0$$

$$\Rightarrow 22w + 22\bar{w} + 33 = 0$$

$$\Rightarrow 22(w+\bar{w}) + 33 = 0$$

or

$$\Rightarrow 44u + 33 = 0 \quad [\because w+\bar{w} = 2u]$$

$4u + 3 = 0$ which is the desired equation.

P-482

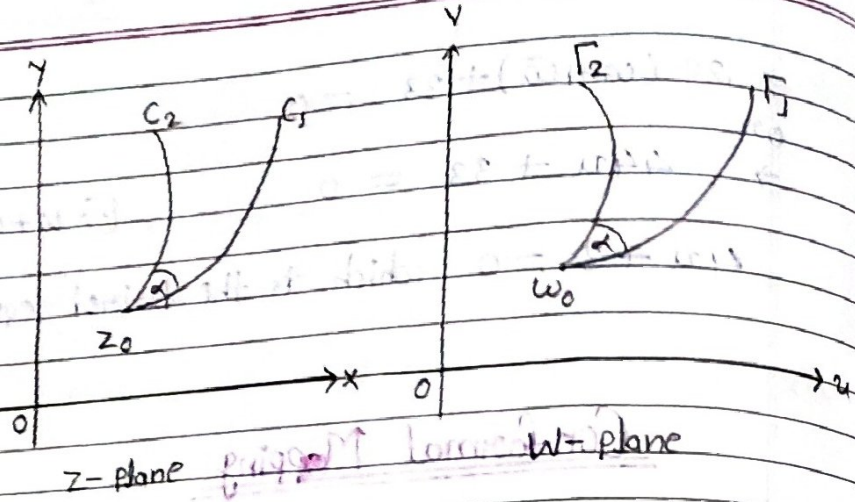
Conformal Mapping

Definition -

Let the transformation

$$w = u + iv \quad \text{and} \quad z = x + iy$$

map a point $P(x_0, y_0)$ of the z -plane to a point $P'(u_0, v_0)$ of the w -plane and let the curves C_1 and C_2 intersecting at Z_0 in z -plane mapped into curve Γ_1 and Γ_2 intersecting at w_0 in w -plane respectively then the transformation is said to be conformal at angle between C_1 and C_2 at Z_0 is equal to angle between Γ_1 and Γ_2 at w_0 .



Conformal Transformation -

If the transformation is such that the angle between C_1 and C_2 at $Z_0 = (x_0, y_0)$ is equal both in magnitude and sense to the angle between Γ_1 and Γ_2 at $w_0 = (u_0, v_0)$ it is said to be conformal at $Z_0 = (x_0, y_0)$.

Sufficient Condition for $W = f(z)$ to Represent a Conformal Mapping

Theorem-(I) Let $f(z)$ be an analytic function of z in a domain D of the z -plane and let $f'(z) \neq 0$ inside D . Then the mapping $w = f(z)$ is conformal at all points of D .

OR

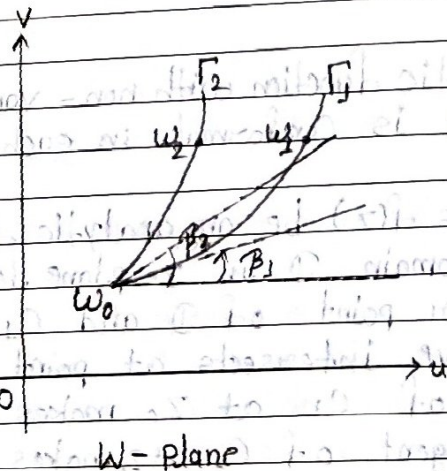
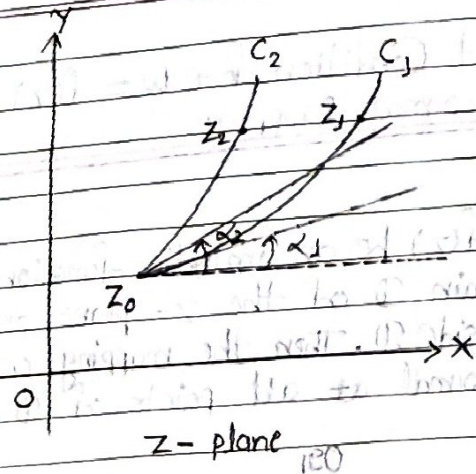
Each analytic function with non-vanishing derivative is conformal in each region.

Proof - Let $W = f(z)$ be an analytic function of z in domain D in z -plane let Z_0 be an interior point of D and C_1 and C_2 be two curve intersects at point Z_0 and let tangent of C_1 at Z_0 makes α_1 angle and tangent of C_2 at Z_0 makes α_2 angle with real axis and Z_1 and Z_2 be two points on the curve C_1 and C_2 respectively at same distance r from Z_0 .

$$Z_1 - Z_0 = re^{j\theta_1}$$

and

$$Z_2 - Z_0 = re^{j\theta_2}$$



As $r \rightarrow 0$, $z_1 \rightarrow z_0$ and $z_2 \rightarrow z_0$ and
and $\lim_{r \rightarrow 0} \theta_1 = \alpha_1$ and $\lim_{r \rightarrow 0} \theta_2 = \alpha_2$

Let w_0, w_1, w_2 are images of point z_0, z_1, z_2 in w -plane and curve Γ_1 and Γ_2 images of C_1 and C_2

Suppose that tangent of Curve Γ_1 at z_0 and Curve Γ_2 at z_0 makes β_1 and β_2 angles with real axis in w -plane

$$w_1 - w_0 = \rho_1 e^{j\phi_1}$$

$$\text{and } w_2 - w_0 = \rho_2 e^{j\phi_2}$$

$$\lim_{\rho_1 \rightarrow 0} \phi_1 = \beta_1 \quad \text{and} \quad \lim_{\rho_2 \rightarrow 0} \phi_2 = \beta_2$$

Since $f(z)$ is analytic i.e. $f(z)$ is differentiable. Then we have

$$f'(z_0) = \lim_{z_1 \rightarrow z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0}$$

$$= \lim_{z_1 \rightarrow z_0} \frac{w_1 - w_0}{z_1 - z_0}$$

$$f'(z_0) = \lim_{z_1 \rightarrow z_0} \frac{\rho_1 e^{j\phi_1}}{r e^{j\theta_1}}$$

$$= \lim_{z_1 \rightarrow z_0} \left(\frac{\rho_1}{r} \right) e^{j(\phi_1 - \theta_1)}$$

Since $f'(z_0)$ is a complex number, so we can write $f'(z_0) = R_0 e^{j\theta_0}$

$$\lim_{z_1 \rightarrow z_0} \left(\frac{p_1}{r} \right) e^{j(\phi_1 - \theta_1)} = R_0 e^{j\theta_0}$$

Now Equating modulus and amplitude from both side

$$R_0 = \lim_{z_1 \rightarrow z_0} \frac{p_1}{r} \text{ and } \theta_0 = \lim_{z_1 \rightarrow z_0} (\phi_1 - \theta_1)$$

$$\lim \phi_1 = \lim \theta_1 + \theta_0 \Rightarrow \beta_1 = \alpha_1 + \theta_0$$

Similarly we can show

$$\beta_2 = \alpha_2 + \theta_0$$

$$\beta_2 - \beta_1 = \alpha_2 + \theta_0 - \alpha_1 - \theta_0$$

$$\beta_2 - \beta_1 = \alpha_2 - \alpha_1$$

Hence $w = f(z)$ is conformal mapping.

Spaces of Analytic Functions

Let G be an open subset of the Complex plane C and let $H(G)$ be the collection of analytic on G and $H(G)$ is subset of space $C(G, C')$ both are said to be space of analytic function $H(G)$ and $C(G, C')$.

Weierstrass's Theorem

P-546

13, 15, 13, 11, 10
21

Theorem (I) Let G be an open subset of C' and let $\{f_n\}$ be a sequence in $H(G)$ and $f \in C(G, C')$ such that $f_n \rightarrow f$. Then f is analytic and $f_n^k \rightarrow f^k$ for each integer $k \geq 1$.

Proof - Given that G is open subset of C' and $\{f_n\}$ is sequence in $H(G)$ and $f \in C(G, C')$ and $\lim_{n \rightarrow \infty} |f_n| = f$ for $\forall z \in D \subseteq C(G, C')$

then we have to show that

$$\lim_{n \rightarrow \infty} f_n^k = f^k$$

Let A is Compact Subset of D and $f_n \rightarrow f$ is uniformly in D then

$$\int_A f dz = \lim_{n \rightarrow \infty} \int_A f_n(z) dz$$

$\therefore f_n(z)$ is analytic in D so $\int_A f_n(z) dz = 0$

$$\int_A f(z) dz = 0$$

Then by morera's theorem $f(z)$ is analytic in $A \subseteq D$

And $G = \bigcup_{n=1}^{\infty} D_n \Rightarrow f$ is analytic in G

Now we will show that $f_n^k \rightarrow f^k$ in G

Let $D = \bar{B}(a, r) \subset G$ then there exists a number $R > r$ such that $\bar{B}(a, R) \subset G$

Let γ be a circle $|z-a|=r$

then by Cauchy integral formula

$$f_n^k(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f_n(w)}{(w-z)^{k+1}} dw$$

$$f^k(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{k+1}} dw$$

Teacher's Signature.....

$$f_n^k(z) - f^k(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f_n(w) - f(w)}{(w-z)^{k+1}} dw$$

$\forall z \in D$ — (1)

Let $M_n = \sup \{ |f_n(w) - f(w)| : |w-a| = R \}$ then for $|w-a| = R$ and $|z-a| \leq r$ we have

$$\begin{aligned} \left| \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \right| &\leq \frac{|f_n(w) - f(w)|}{|w-z|^{k+1}} \\ &= \frac{|f_n(w) - f(w)|}{|w-a - (z-a)|^{k+1}} \\ &\leq \frac{|f_n(w) - f(w)|}{[|w-a| - |z-a|]^{k+1}} \end{aligned}$$

$$\left| \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \right| \leq \frac{|f_n(w) - f(w)|}{(R-r)^{k+1}}$$

$$\left| \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \right| \leq \frac{\sup \{ |f_n(w) - f(w)| : |w-a| = R \}}{(R-r)^{k+1}}$$

$$\therefore M_n = \sup \{ |f_n(w) - f(w)| : |w-a| = R \}$$

Teacher's Signature.....

$$\frac{f_n(w) - f(w)}{(w-z)^{k+1}} \leq \frac{M_n}{(R-Y)^{k+1}} \quad \text{--- (3)}$$

from (1)

$$|f_n^k(z) - f^k(z)| = \left| \frac{k!}{2\pi i} \int \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \right|$$

$$\leq \frac{k!}{2\pi} \int \left| \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \right| |dw|$$

using (3)

$$|f_n^k(z) - f^k(z)| \leq \frac{k!}{2\pi} \int \frac{M_n}{(R-Y)^{k+1}} |dw|$$

$$|f_n^k(z) - f^k(z)| \leq \frac{k!}{2\pi} \frac{M_n}{(R-Y)^{k+1}} \cdot 2\pi R$$

$$\because \int_Y |dw| = 2\pi R$$

When $n \rightarrow \infty$ $M_n \rightarrow 0$ $\therefore f_n \rightarrow f$

Then it follows that

$$|f_n^k - f^k(z)| < \epsilon \quad \forall n \geq m$$

This $f_n^k(z)$ is uniformly converges to $f^k(z)$ for each $k \geq 1$.

Teacher's Signature

P. 547

17, 15, 14, 12, 19, 10, 9, 11

Hurwitz's Theorem

Theorem (ii) Let $\{f_n(z)\}$ be a sequence of functions, each analytic in a domain D bounded by a closed contour, and let $f_n(z)$ converges to $f(z)$ uniformly in D . If f_n have no zeros in D then either $f \equiv 0$ or f has no zeros in D .

or

Proof - Give and prove that Hurwitz's theorem for the space of analytic functions.

Proof - Given that $\{f_n(z)\}$ be a sequence of functions in domain D and each function is analytic in D and $\{f_n(z)\}$ is converges to $f(z)$ in D . Then we will show that either $f_n(z)$ have no zero in D or $f = 0$.

Suppose that $f \neq 0$ then we will show that $f(z)$ has no zero in D . Supposing contrary that

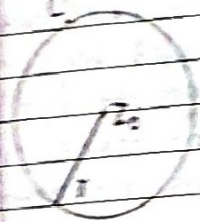
$$f(z_0) = 0 \quad \text{for some } z_0 \in D$$

Since zeros of an analytic function are isolated so we can choose $\delta > 0$ such that

$$A = \{z : 0 < |z - z_0| \leq \delta\} \subseteq D$$

and $f(z)$ is analytic and $f(z) \neq 0$ on A .

$\Rightarrow f(z) \neq 0$ in circle $|z - z_0| = r$



and let circle $|z - z_0| = r$ is denoted by C then

$\Rightarrow \frac{1}{f_n(z)}$ converges uniformly to $\frac{1}{f(z)}$ on C and $f_n(z)$ and $f(z)$

both are analytic so

$\Rightarrow f'_n(z)$ converges uniformly to $f'(z)$ on C

$\Rightarrow \frac{f'_n(z)}{f_n(z)}$ converges to $\frac{f'(z)}{f(z)}$ on C

$$\Rightarrow \int_C \frac{f'(z)}{f(z)} dz = \lim_{n \rightarrow \infty} \int_C \frac{f'_n(z)}{f_n(z)} dz \quad \text{--- (1)}$$

Since f_n have no zero in D then by Cauchy Goursat theorem

$$\int_C \frac{f'_n(z)}{f_n(z)} dz = 0$$

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = 0 \quad \text{--- (2)}$$

Teacher's Signature.....

but we know that by argument principal

$$N = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz \quad \text{--- (3)}$$

where N is number of zero inside C

its clear that from (2) and (3)

$\Rightarrow f$ has no zero inside C

$\Rightarrow f$ has no zero inside D

$\Rightarrow f(z_0) \neq 0$ in D .

Teacher's Signature.....

Montel's Theorem

Theorem-(iii) Statement - A family F of holomorphic functions defined in a domain D , that is $F \subset H(D)$ is normal if and only if F is locally bounded.

or

State and prove Montel's theorem.

Proof - Given that F is holomorphic function in domain D .

If part - Suppose that $F \subset H(D)$ is normal. Then for given compact subset $K \subset D$

Then we will prove that

$$|f(z)| \leq M \quad \forall f \in F \text{ in } K \subseteq D$$

Since F is normal then by Arzela - Ascoli theorem

$\Rightarrow F$ is equicontinuous for given $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$|z-w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon \quad \forall z, w \in K \quad (1)$$

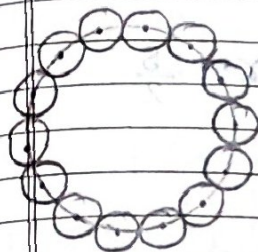
Teacher's Signature.....

$\Rightarrow F$ is equicontinuous in $K \subseteq G$

Then by definition of equicontinuous for given $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$|z-w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon$$

$$\forall z, w \in K \quad (1)$$



Since K is compact then definition of compactness $\exists$ finitely open covers $B(z_j, r)$ such that

$$K \subset \bigcup_{j=1}^K B(z_j, r) \quad (2)$$

where $z_1, z_2, \dots, z_K \in K$

Since F is normal $\Rightarrow$ for every fixed point $z \in D$

set $A_z = \{f(z) : f \in F\}$ be contained in compact subset and since every compact subset is bounded

$\Rightarrow \exists$ a constant M_0 such that

$$|f(z_j)| < M_0 \quad \forall f \in F \text{ and } z_j \in K \quad (3)$$

$\forall j \in \{1, 2, \dots, K\}$

Teacher's Signature.....

For any $z \in K$ using (2)
 $\Rightarrow z \in \bigcup_{j=1}^k B(z_j, r)$

$\Rightarrow z \in B(z_j, r)$ for some $1 \leq j \leq k$

$$\Rightarrow |z - z_j| < r$$

$$\text{Let } r = \frac{\delta}{2}$$

such that

$$|z - z_j| < \frac{\delta}{2}$$

$$|z - z_j| < \delta$$

using (1)

$$|z - z_j| < \delta \Rightarrow |f(z) - f(z_j)| < \varepsilon \quad (4)$$

Thus for $z \in K$

$$|f(z)| = |f(z) - f(z_j) + f(z_j)|$$

$$|f(z)| \leq |f(z) - f(z_j)| + |f(z_j)|$$

using (4) and (3)

$$|f(z)| < \varepsilon + M_0$$

$$\varepsilon + M_0 = M, \text{ say.}$$

$$|f(z)| < M$$

Hence F is locally bounded.

Teacher's Signature.....

Conversely -

In this part given that $f(z)$ is bounded then we will prove that F is normal.

To prove F is normal we will prove F is equicontinuous at each point in D for given $\varepsilon > 0$ and each point $a \in D$ there is an $r > 0$ such that

$$B(a, r) \subset D$$

Let $w, w' \in B(a, \frac{r}{2})$ then by Cauchy integral formula

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - w} d\xi$$

$$f(w') = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - w'} d\xi$$

$$f(w) - f(w') = \frac{1}{2\pi i} \int_C \left[\frac{1}{\xi - w} - \frac{1}{\xi - w'} \right] f(\xi) d\xi$$

$$f(w) - f(w') = \frac{1}{2\pi i} \int_C \frac{(\xi - w') - (\xi - w)}{(\xi - w)(\xi - w')} f(\xi) d\xi$$

$$f(w) - f(w') = \frac{1}{2\pi i} \int_C \frac{(w - w')}{(\xi - w)(\xi - w')} f(\xi) d\xi$$

Teacher's Signature.....

$$f(w) - f(w') = \frac{1}{2\pi i} \int_C \frac{(w-w')f(z)dz}{(z-w)(z-w')}$$

$$|f(w) - f(w')| \leq \frac{1}{2\pi} \int_C \frac{|w-w'| |f(z)| |dz|}{|z-w| |z-w'|}$$

$$\therefore |f(z)| \leq M \quad \forall z \in D$$

$$|f(w) - f(w')| \leq \frac{M|w-w'|}{2\pi} \int_C \frac{|dz|}{|z-a+a-w| |z-a+a-w'|}$$

$$= \frac{M|w-w'|}{2\pi} \int_C \frac{|dz|}{|(z-a)-(w-a)| |(z-a)-(w'-a)|}$$

$$\leq \frac{M|w-w'|}{2\pi} \int_C \frac{|dz|}{|z-a|^2}$$

Putting $z = a + re^{j\theta}$

$$dz = re^{j\theta} d\theta$$

$$= \frac{M|w-w'|}{2\pi} \int_0^{2\pi} \frac{r e^{j\theta} d\theta}{|a+re^{j\theta}-a| |w-a| |a+re^{j\theta}-a|}$$

$$= \frac{M|w-w'|}{2\pi} \int_0^{2\pi} \frac{r e^{j\theta} d\theta}{|re^{j\theta}-w-a| |re^{j\theta}-w'-a|}$$

put $\therefore |e^{j\theta}| = 1$

$$|w-a| < \frac{r}{2}$$

$$\text{and } |w'-a| < \frac{r}{2}$$

$$\leq \frac{M|w-w'|}{2\pi} \int_0^{2\pi} \frac{r d\theta}{(r-\frac{r}{2})(r-\frac{r}{2})}$$

$$\leq \frac{4M|w-w'|}{2\pi r} \int_0^{2\pi} d\theta$$

$$|f(w) - f(w')| \leq \frac{4M|w-w'|}{2\pi r} \cdot 2\pi$$

$$|f(w) - f(w')| \leq \frac{4M|w-w'|}{r} \quad \text{--- (5)}$$

Since K is compact set then $\exists$ a open cover

$$B(z_j, \frac{r_j}{4}) \quad \forall j = 1, 2, \dots, k$$

and $|f(z_j)| \leq M_j \quad \forall f \in F$
on $B(z_j, r_j)$

And let $M = \max \{M_j : j = 1, 2, \dots, k\}$
 $\gamma = \min \{\gamma_j : j = 1, 2, \dots, k\}$

and $\delta = \min \left\{ \frac{\gamma}{2}, \frac{\gamma \epsilon}{4M} \right\}$

$|w - w'| < \delta$ in K

and $w \in B(z_j, \frac{\gamma_j}{4})$

$|w' - z_j| = |w' - w + w - z_j|$

$\leq |w' - w| + |w - z_j|$

$\therefore |w - w'| < \delta$

$|w' - z_j| < \delta + \frac{\gamma_j}{4}$

$< \frac{\gamma_j}{2} + \frac{\gamma_j}{4}$

$|w' - z_j| < \frac{3\gamma_j}{4} < \gamma_j$

$w' \in B(z_j, \gamma_j)$

$\Rightarrow w, w' \in B(z_j, \gamma_j)$

from (5)

$|f(w) - f(w')| < \frac{4M\delta}{\gamma}$

let $4M\delta < \epsilon$

$|f(w) - f(w')| < \epsilon$

finally

$|w - w'| < \delta \Rightarrow |f(w) - f(w')| < \epsilon$

Hence F is equicontinuous then by Ascoli-Arzelà theorem F is normal.

Lemma-1. Let the metric p be defined as in (1). If $\varepsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset G$ such that for $f, g \in (G, \Omega)$,

$$\sup \{d(f(z), g(z)) : z \in K\} < \delta \Rightarrow p(f, g) < \varepsilon$$

Conversely if $\delta > 0$ and a compact set K are given then there is an $\varepsilon > 0$ such that for $f, g \in (G, \Omega)$

$$p(f, g) < \varepsilon \Rightarrow \sup \{d(f(z), g(z)) : z \in K\} < \delta$$

Proof-